A lattice-theoretic interpretation of independence of frames

Fabio Cuzzolin

INRIA Rhône-Alpes, 655 avenue de l'Europe, 38334 SAINT ISMIER CEDEX, France `Fabio.Cuzzolin@inrialpes.fr`

Summary. In this paper we discuss the nature of independence of sources in the theory of evidence from an algebraic point of view, starting from the analogy with the case of projective geometries. Independence in Dempster's rule is equivalent to independence of frames as Boolean algebras. Collection of frames, though, can be interpreted as semi-modular lattices on which independence can be defined in several different forms. We prove those forms to be distinct but related to Boolean independence, as a step towards a more general definition of this fundamental notion.

Key words: *Dempster's rule, frames, semi-modular lattice, independence.*

1 Introduction

The theory of evidence was born as a contribution to a mathematically rigorous description of subjective probability, where different observers (or "experts") of the same phenomenon possess in general different notions of what the decision space is. Mathematically, this translates into admitting the existence of several distinct representations of the decision space at different levels of refinement. Evidence available on those spaces or *frames* can then be "moved" to a common frame or "common refinement" to be fused. In the theory of evidence, information fusion takes place by combining evidence in the form of belief functions by means of Dempster's orthogonal sum [5].
Dempster's combination, however, is guaranteed to exist [4] only when the original frames are *independent* [15]. Combinability (in Dempster's approach) and independence of frames are strictly intertwined.

Evidence combination has indeed been widely studied [24, 23] in different mathematical frameworks [19, 7]: An exhaustive review would be impossible here [1, 11, 12, 2, 14]. In particular, a lot of work has been done on the issue of merging conflicting evidence [6, 8, 10, 22], while some attention has been given to situations in which the latter comes from dependent sources [3]. On the other hand not much work has been done on the properties of the families of compatible frames [17, 9, 4].

Here we build on the results obtained in [4] to complete an algebraic analysis of families of frames and conduct a comparative study of the notion of independence, so central in the theory of evidence, in an algebraic setup.
First, we recall the fundamental result on the equivalence between independence of sources in Dempster's combination (Section 2) and independence of frames (Section 2.3). In this incarnation independence of sources can indeed be studied from an algebraic point of view, and compared with other classical forms of independence (Section 2.4)). In the core of the paper (Section 3) we prove in particular that families of compatible frames form *semi-modular lattices*, extending some recent preliminary results [4]. Independence can be defined on semi-modular lattices in several different forms: We can then study the relationship between evidential and lattice independence in all those different formulations (Section 4): they turn out to be distinct, but nevertheless strictly related.
As independence of frames is a direct consequence of independence of Boolean sub-algebras [18], the overall picture opens the way to a more comprehensive definition of this basilar concept.

2 Independence of sources in Dempster's combination

2.1 Dempster's combination of belief functions

In the theory of evidence a *basic probability assignment* (b.p.a.) over a finite set or *frame* [15] Θ is a function $m : 2^\Theta \rightarrow [0,1]$ on its power set $2^\Theta = \{A \subseteq \Theta\}$ such that $m(\emptyset) = 0$, $\sum_{A\subseteq\Theta} m(A) = 1$, $m(A) \geq 0 \ \forall A \subseteq \Theta$.
The *belief function* (b.f.) $b : 2^\Theta \rightarrow [0,1]$ associated with a b.p.a. m on Θ is defined as $b(A) = \sum_{B\subseteq A} m(B)$. The *orthogonal sum* or *Dempster's sum* of two b.f.s b_1, b_2 is a new belief function $b_1 \oplus b_2$ with b.p.a.

$$m_{b_1\oplus b_2}(A) = \frac{\sum_{B\cap C=A} m_{b_1}(B)m_{b_2}(C)}{\sum_{B\cap C\neq\emptyset} m_{b_1}(B)m_{b_2}(C)}. \tag{1}$$

When the denominator of (1) is zero the two b.f.s are *non-combinable*.

2.2 Independence of sources

Independence plays a central role in Dempster's combination (1), as it is the fundamental assumption under which the combination of two belief functions can actually take place.
Consider a problem in which we have probabilities for a question Q_1 and we want to obtain degrees of belief for a related question Q_2 [16, 20], with Ω and Θ the sets of possible answers to Q_1 and Q_2 respectively. Formally, given a

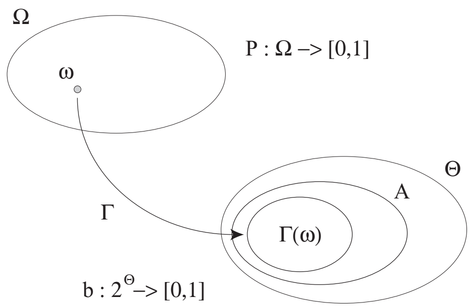

Fig. 1. A probability measure P on Ω induces a belief function b on Θ whose values on the events A of Θ are given by $b(A) = \sum_{\omega\in\Omega:\Gamma(\omega)\subseteq A} P(\omega)$.

probability measure P on Ω we want to derive a degree of belief $b(A)$ that $A \subseteq \Theta$ contains the correct response to Q_2 (see Figure 1). Let us call $\Gamma(\omega)$ the subset of answers to Q_2 compatible with a given outcome $\omega \in \Omega$ of Q_1. The map $\Gamma : \Omega \to 2^\Theta$ is called a *multi-valued mapping*.
The degree of belief $b(A)$ of an event $A \subseteq \Theta$ is then the total probability of all answers ω that satisfy the above condition, namely $b(A) = P(\{\omega|\Gamma(\omega) \subseteq A\})$.
Consider now two multi-valued mappings Γ_1, Γ_2 inducing two b.f.s b_1, b_2 on the same frame Θ, Ω_1 and Ω_2 their domains and P_1, P_2 the associated probability measures on Ω_1 and Ω_2, respectively. If we suppose that the items of evidence generating P_1 and P_2 are *independent*, we are allowed to build the product space $(\Omega_1 \times \Omega_2, P_1 \times P_2)$: the detection of two outcomes $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ will then tell us that the answer to Q_2 is somewhere in $\Gamma(\omega_1, \omega_2) = \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2)$. We then need to condition the product measure $P_1 \times P_2$ over the set of pairs (ω_1, ω_2) whose images have non-empty intersection: $P = P_1 \times P_2|_\Omega$, with $\Omega = \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2|\Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \neq \emptyset\}$. This new belief function b is precisely the orthogonal sum of b_1, b_2.

2.3 Independence of sources and independence of frames

Families of compatible frames

Dempster's mechanism for evidence combination then assumes that the domains on which the evidence is present (in the form of a probability measure) are independent. This concept is mirrored by the notion of *independence of compatible frames* [15]. Given two frames Θ and Θ', a map $\rho : 2^\Theta \to 2^{\Theta'}$ is a *refining* if ρ maps the elements of Θ to a disjoint partition of Θ': $\rho(\{\theta\}) \cap \rho(\{\theta'\}) = \emptyset \ \forall \theta, \theta' \in \Theta$, $\bigcup_{\theta\in\Theta} \rho(\{\theta\}) = \Theta'$, with $\rho(A) = \cup_{\theta\in A}\rho(\{\theta\})$ $\forall A \subset \Theta$. Θ' is called a *refinement* of Θ, Θ a *coarsening* of Θ'.

Shafer calls a structured collection of frames a *family of compatible frames of discernment* ([15], pages 121-125). In particular, in such a family every pair of frames has a common refinement, i.e. a frame which is a refinement of both.

If $\Theta_1, ..., \Theta_n$ are elements of a family of compatible frames $\mathcal{F}$ then there exists a *unique* common refinement $\Theta \in \mathcal{F}$ of them such that $\forall \theta \in \Theta\ \exists \theta_i \in \Theta_i$ for $i = 1, ..., n$ such that

$$\{\theta\} = \rho_1(\theta_1) \cap \cdots \cap \rho_n(\theta_n),$$

where ρ_i denotes the refining between Θ_i and Θ. This unique frame is called the *minimal refinement* $\Theta_1 \otimes \cdots \otimes \Theta_n$ of $\Theta_1, ..., \Theta_n$.

In the example of Figure 2 we want to find out the position of a target point in an image. We can pose the problem on a frame $\Theta_1 = \{c_1, ..., c_5\}$

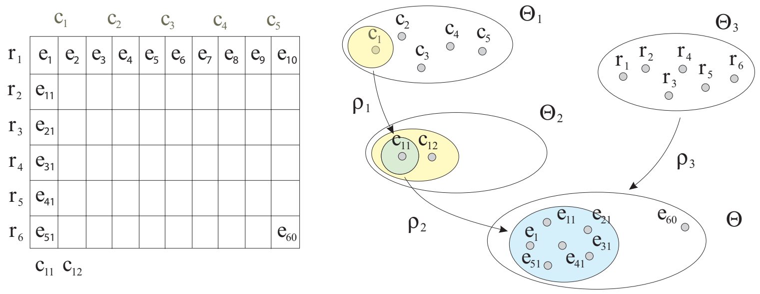

Fig. 2. An example of family of compatible frames. Different discrete quantizations of row and column ranges of an image have a common refinement, the set of cells shown on the left. The refinings ρ_1, ρ_2, ρ_3 between those frames appear to the right.

obtained by partitioning the column range of the image into 5 intervals, but we can also partition it into 10 intervals, obtaining a different frame $\Theta_2 = \{c_{11}, c_{12}, ..., c_{51}, c_{52}\}$. The row range can also be divided in, say, 6 intervals $\Theta_3 = \{r_1, ..., r_6\}$. All those frames belong to a family of compatible frames, with the collection of cells $\Theta = \{e_1, ..., e_{60}\}$ depicted in Figure 2-left as common refinement, and refinings shown in Figure 2-right. It is easy to verify that Θ is the minimal refinement of Θ_2, Θ_3 as, for example, $\{e_{41}\} = \rho_2(c_{11}) \cap \rho_3(r_4)$.

Independence of frames

Now, let $\Theta_1, ..., \Theta_n$ be elements of a family of compatible frames, and $\rho_i : \Theta_i \rightarrow 2^{\Theta_1 \otimes \cdots \otimes \Theta_n}$ the corresponding refinings to their minimal refinement. $\Theta_1, ..., \Theta_n$ are *independent* [15] ($\mathcal{IF}$) if, whenever $\emptyset \neq A_i \subseteq \Theta_i$ for $i = 1, ..., n$,

$$\rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset. \tag{2}$$

In particular, if $\exists j \in [1, .., n]$ s.t. Θ_j is a coarsening of some other frame Θ_i, $\Theta_1, ..., \Theta_n$ are *not* $\mathcal{IF}$. An equivalent condition is [4]

$$\Theta_1 \otimes \cdots \otimes \Theta_n = \Theta_1 \times \cdots \times \Theta_n \tag{3}$$

i.e. their minimal refinement is their Cartesian product.

Now, independence of frames and Dempster's rule are strictly related [4].

Proposition 1. *Let $\Theta_1, ..., \Theta_n$ elements of a family of compatible frames. Then they are independent iff all the possible collections of b.f.s $b_1, ..., b_n$ defined respectively on $\Theta_1, ..., \Theta_n$ are combinable on their minimal refinement $\Theta_1 \otimes \cdots \otimes \Theta_n$.*

Proposition 1 states that independence of frames and independence of sources (which is at the root of Dempster's combination) are in fact equivalent.

This is not at all surprising when we compare the condition under which Dempster's sum is well defined

$$\Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \neq \emptyset, \qquad (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$$

with independence of frames expressed as

$$\rho_1(\theta_1) \cap \rho_2(\theta_2) \neq \emptyset, \qquad (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2.$$

2.4 An algebraic study of independence

In its equivalent form of independence of frames (Proposition 1) independence of sources can be analyzed from an algebraic point of view.
A powerful intuition comes from the intriguing similarity between $\mathcal{IF}$ and independence of vector subspaces (recalling Equations 2 and 3):

$$\begin{array}{ccc} \rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset, \ \forall A_i \subseteq \Theta_i & \equiv & \Theta_1 \otimes \cdots \otimes \Theta_n = \Theta_1 \times \cdots \times \Theta_n \\ v_1 + \cdots + v_n \neq 0, \ \forall v_i \in V_i & \equiv & span\{V_1, ..., V_n\} = V_1 \times \cdots \times V_n. \end{array} \tag{4}$$

While a number of compatible frames $\Theta_1, ..., \Theta_n$ are $\mathcal{IF}$ iff each choice of their representatives $A_i \in 2^{\Theta_i}$ has non-empty intersection, a collection of vector subspaces $V_1, ..., V_n$ is "independent" iff for each choice of vectors $v_i \in V_i$ the sum of those vectors is non-zero. These relations, introduced in what seem very different contexts, can be formally obtained from each other under the following correspondence of quantities and operators:

$$v_i \leftrightarrow A_i, \quad V_i \leftrightarrow 2^{\Theta_i}, \quad + \leftrightarrow \cap, \quad 0 \leftrightarrow \emptyset, \quad \otimes \leftrightarrow span.$$

As we will see here, families of frames and collections of subspaces of a vector space or "projective geometries" share the algebraic structure of *semi-modular lattice*, which in turn admits a characteristic notion of independence. It is natural to wonder how $\mathcal{IF}$ is related to lattice-theoretic independence.

3 The semi-modular lattice of frames

3.1 Lattices

A *partially ordered set* or *poset* is a set P endowed with a binary relation $\leq$ such that, for all x, y, z in P the following conditions hold: 1. $x \leq x$; 2. if

$x \leq y$ and $y \leq x$ then $x = y$; 3. if $x \leq y$ and $y \leq z$ then $x \leq z$. In a poset we say that x "covers" y ($x \succ y$) if $x \geq y$ and there is no intermediate element in the chain linking them. A classical example is the power set 2^Θ of a set Θ together with the set-theoretic inclusion $\subset$. Given two elements $x, y \in P$ of a poset P their *least upper bound* $x \vee y$ is the smallest element of P that is bigger than both x and y, while their *greatest lower bound* $x \wedge y$ is the biggest element of P that is smaller than both x and y. Not every pair of elements of a poset, though, is guaranteed to admit inf and/or sup.
A *lattice* L is a poset in which each *pair* of elements admits both inf and sup. When each *arbitrary* (even not finite) collection of elements of L admits both inf and sup, L is said *complete*. In this case there exist $\mathbf{0} \equiv \wedge L$, $\mathbf{1} \equiv \vee L$ called respectively *initial* and *final* element of L. 2^Θ is complete, with $\mathbf{0} = \emptyset$ and $\mathbf{1} = \{\Theta\}$. The *height* $h(x)$ of an element x in L is the length of a maximal chain from $\mathbf{0}$ to x. In the case of the power set 2^Θ, the height of a subset $A \in 2^\Theta$ is simply its cardinality $|A|$.

3.2 Semi-modularity of the lattice of frames

In a family of compatible frames one can define the following order relation:

$$\Theta_1 \leq \Theta_2 \Leftrightarrow \exists \rho : \Theta_2 \to 2^{\Theta_1} \; refining \tag{5}$$

i.e. Θ_1 is a refinement of Θ_2. The inverse relation $\Theta_1 \leq^* \Theta_2$ iff Θ_1 is a coarsening of Θ_2 is also a valid ordering. After introducing the notion of *maximal coarsening* as the largest cardinality common coarsening $\Theta_1 \oplus \cdots \oplus \Theta_n$ of a given collection of frames $\Theta_1, \cdots, \Theta_n$, we can prove that [4]

Proposition 2. *Both* $(\mathcal{F}, \leq)$ *and* $(\mathcal{F}, \leq^*)$ *where* $\mathcal{F}$ *is a family of compatible frames are lattices, with respectively* $\bigwedge_i \Theta_i = \bigotimes_i \Theta_i$, $\bigvee_i \Theta_i = \bigoplus_i \Theta_i$ *and* $\bigwedge_i^* \Theta_i = \bigoplus_i \Theta_i$, $\bigvee_i^* \Theta_i = \bigotimes_i \Theta_i$.

A special class of lattices arises from *projective geometries*, i.e. collections $L(V)$ of all subspaces of a given vector space V.

Definition 1. *A lattice* L *is* upper semi-modular *if for each pair* x, y *of elements of* L, $x \succ x \wedge y$ *implies* $x \vee y \succ y$. *A lattice* L *is* lower semi-modular *if for each pair* x, y *of elements of* L, $x \vee y \succ y$ *implies* $x \succ x \wedge y$.

Clearly if L is upper semi-modular with respect to an order relation $\leq$, the corresponding dual lattice with order relation $\leq^*$ is lower semi-modular.

Theorem 1. $(\mathcal{F}, \leq)$ *is an upper semi-modular lattice;* $(\mathcal{F}, \leq^*)$ *is a lower semi-modular lattice.*

Proof. We just need to prove the upper semi-modularity of $\mathcal{F}$ with respect to $\leq$. Consider two compatible frames Θ, Θ', and suppose that Θ covers their minimal refinement $\Theta \otimes \Theta'$ (their inf with respect to $\leq$). The proof articulates into the following steps (see Figure 3):

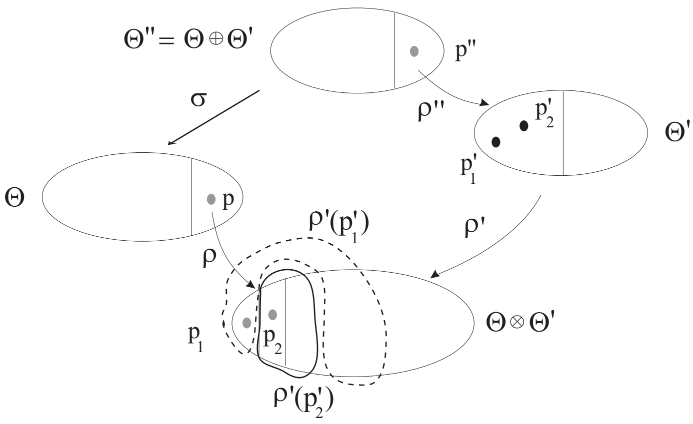

Fig. 3. Upper semi-modularity of $(\mathcal{F}, \leq)$.

- since Θ covers $\Theta \otimes \Theta'$ we have that $|\Theta| = |\Theta \otimes \Theta'| + 1$;
- hence there exists a single element $p \in \Theta$ which is refined into two elements p_1, p_2 of $\Theta \otimes \Theta'$: all other elements of Θ are left unchanged: $\{p_1, p_2\} = \rho(p)$;
- this in turn implies that p_1, p_2 belong each to the image of a different element of Θ' (otherwise Θ would itself be a refinement of Θ', and we would have $\Theta \otimes \Theta' = \Theta$): $p_1 \in \rho'(p'_1)$, $p_2 \in \rho'(p'_2)$;
- if we merge p'_1, p'_2 we have a coarsening Θ'' of Θ': $\{p'_1, p'_2\} = \rho''(p'')$;
- but Θ'' is a coarsening of Θ, too, as we can build the following refining

$$\sigma : \Theta'' \to 2^{\Theta} : \sigma(q) = \rho'(\rho''(q))$$

  as $\rho'(\rho''(q))$ is a subset of Θ $\forall q \in \Theta''$:
  - if $q = p''$, $\sigma(q)$ is $\{p\} \cup (\rho'(p'_1) \setminus \{p_1\}) \cup (\rho'(p'_2) \setminus \{p_2\})$;
  - if $q \neq p''$, $\rho'(\rho''(q))$ is also a set of elements of Θ, as all elements of Θ but p are left unchanged by ρ;
- as $|\Theta''| = |\Theta'| - 1$, Θ'' is the maximal coarsening of Θ, Θ': $\Theta'' = \Theta \oplus \Theta'$;
- hence Θ' covers $\Theta \oplus \Theta'$, which is the sup of Θ, Θ' in $(\mathcal{F}, \leq)$.

Theorem 1 strengthens the main result of [4], where we proved that finite families of frames are Birkhoff. A lattice is *Birkhoff* if $x \wedge y \prec x, y$ implies $x, y \prec x \vee y$. (Upper) semi-modularity implies the Birkhoff property, but not vice-versa.

3.3 Finite lattice of frames

We will here focus on *finite* families of frames. Given a set of compatible frames $\Theta_1, ..., \Theta_n$ consider the set $P(\Theta)$ of all partitions of their minimal refinement $\Theta = \Theta_1 \otimes \cdots \otimes \Theta_n$. As $\mathcal{IF}$ involves only partitions of $\Theta_1 \otimes \cdots \otimes \Theta_n$, we can conduct our analysis there. We denote by

$$L(\Theta) \doteq (P(\Theta), \leq), \qquad L^*(\Theta) \doteq (P(\Theta), \leq^*)$$

the two lattices associated with $P(\Theta)$. Consider for example the partition

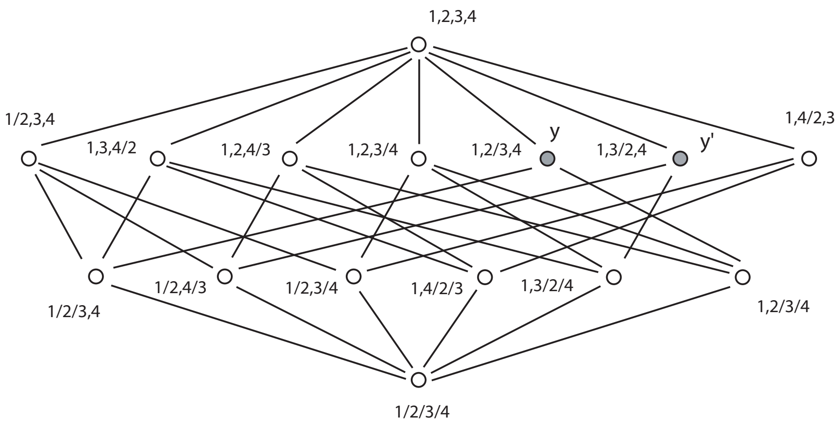

Fig. 4. The partition (lower) semi-modular lattice $L^*(\Theta)$ for a frame Θ of size 4. Partitions $A_1, ..., A_k$ of Θ are denoted by $A_1/.../A_k$. Partitions with the same number of elements are arranged on the same level.

lattice associated with a frame of size 4: $\Theta = \{1,2,3,4\}$, depicted in Figure 4. According to the ordering $\leq^*$ each edge indicates that the bottom element is bigger than the top one. If we pick the pair of partitions $y = \{1,2/3,4\}$ and $y' = \{1,3/2,4\}$, we can notice that both y, y' cover their inf $y \wedge^* y' = \{1,2,3,4\}$ but their sup $y \vee^* y' = \{1/2/3/4\}$ does not cover any of them. Hence, $(P(\Theta), \leq^*)$ is not upper semi-modular but lower semi-modular.

3.4 A lattice-theoretic interpretation of independence

We can now reinterpret the analogy introduced in Section 2.4 between subspaces of a vector space V and elements of a family of compatible frames. Both are lattices: according to the chosen order relation we get an upper $L(\Theta)$ or lower $L^*(\Theta)$ semi-modular lattice (see table)

lattice	$L(V)$	$L^*(\Theta)$	$L(\Theta)$
initial element $\mathbf{0}$	$\{0\}$	$\mathbf{0}_{\mathcal{F}}$	Θ
sup $l_1 \vee l_2$	$span(V_1, V_2)$	$\Theta_1 \otimes \Theta_2$	$\Theta_1 \oplus \Theta_2$
inf $l_1 \wedge l_2$	$V_1 \cap V_2$	$\Theta_1 \oplus \Theta_2$	$\Theta_1 \otimes \Theta_2$
order relation $l_1 \leq l_2$	$V_1 \subseteq V_2$	Θ_1 coars. of Θ_2	Θ_1 refin. of Θ_2
height $h(l_1)$	$\dim(V_1)$	$\|\Theta_1\| - 1$	$\|\Theta\| - \|\Theta_1\|$

where $\mathbf{0}_{\mathcal{F}}$ denotes the unique frame of a family $\mathcal{F}$ with cardinality 1.

4 Independence on lattices and independence of frames

4.1 Independence on lattices

As a matter of fact, abstract independence can be defined on the elements of a semi-modular lattice [21]. Consider again the classical example of linear independence of vectors. By definition $v_1, ..., v_n$ are *linearly independent* iff $\sum_i \alpha_i v_i = 0 \vdash \alpha_i = 0 \ \forall i$: Well known equivalent conditions are:

$$\begin{array}{lll} \mathcal{I}_1 : & v_j \not\subset span(v_i, i \neq j) & \forall j = 1, ..., n; \\ \mathcal{I}_2 : & v_j \cap span(v_1, ..., v_{j-1}) = 0 & \forall j = 2, ..., n; \\ \mathcal{I}_3 : & \dim(span(v_1, ..., v_n)) = n. & \end{array} \tag{6}$$

As 1D subspaces are elements of a lattice $L(V)$ for which $span = \vee$, $\cap = \wedge$, $dim = h$ and $\mathbf{0} = 0$ we can generalize the relations (6) to collections $\{l_1, ..., l_n\}$ of non-zero elements of any semi-modular lattice with initial element $\mathbf{0}$ as

$$\begin{array}{lll} \mathcal{I}_1 : & l_j \not\leq \bigvee_{i \neq j} l_i & \forall j = 1, ..., n; \\ \mathcal{I}_2 : & l_j \wedge \bigvee_{i<j} l_i = \mathbf{0} & \forall j = 2, ..., n; \\ \mathcal{I}_3 : & h(\bigvee_i l_i) = \sum_i h(l_i). & \end{array} \tag{7}$$

4.2 Lattice-theoretic independence on the lattice of frames

Independence assumes then several different forms in lattice theory. As compatible frames form semi-modular lattices it is natural to suppose that some of those may indeed coincide with Shafer's independence of frames, or at least have some relations with it.
We analyze the relations (7) in the flag lower semi-modular case $L^*(\Theta)$:

$$\begin{array}{lll} \Theta_1, ..., \Theta_n \ \mathcal{I}_1^* \Leftrightarrow & \Theta_j \oplus \bigotimes_{i \neq j} \Theta_i \neq \Theta_j & \forall \ j = 1, ..., n \\ \Theta_1, ..., \Theta_n \ \mathcal{I}_2^* \Leftrightarrow & \Theta_j \oplus \bigotimes_{i=1}^{j-1} \Theta_i = \mathbf{0}_{\mathcal{F}} & \forall \ j = 2, ..., n \\ \Theta_1, ..., \Theta_n \ \mathcal{I}_3^* \Leftrightarrow & \Big| \bigotimes_{i=1}^{n} \Theta_i \Big| - 1 = \sum_{i=1}^{n} (|\Theta_i| - 1) & \end{array} \tag{8}$$

as

$$\Theta_i \wedge \Theta_j = \Theta_i \oplus \Theta_j, \quad \Theta_i \vee \Theta_j = \Theta_i \otimes \Theta_j, \quad h^*(\Theta_i) = |\Theta_i| - 1.$$

The frames $\Theta_1, ..., \Theta_n$ are $\mathcal{I}_1^*$ iff none of them is a coarsening of the minimal refinement of all the others; they are $\mathcal{I}_2^*$ iff $\forall j > 1 \ \Theta_j$ does not have a non-trivial common coarsening with the minimal refinement of its predecessors. $\mathcal{I}_3^*$ on its side has a very interesting semantics in terms of probability spaces: As the dimension of the polytope of probability measures definable on a domain of size k is $k - 1$, $\mathcal{I}_3^*$ is equivalent to say that the dimension of the probability polytope for the minimal refinement is the sum of the dimensions of the polytopes associated with the individual frames.

4.3 Evidential independence is stronger than $\mathcal{I}_1^*$, $\mathcal{I}_2^*$

To study the logical implications between these lattice-theoretic relations and independence of frames we first need to prove an interesting Lemma.

Lemma 1. *$\Theta_1, ..., \Theta_n$ $\mathcal{IF}$, $n > 1 \vdash \bigoplus_{i=1}^n \Theta_i = \mathbf{0}_{\mathcal{F}}$.*

Proof. We prove Lemma 1 by induction. For $n = 2$, let us suppose that Θ_1, Θ_2 are $\mathcal{IF}$. Then $\rho_1(A_1) \cap \rho_2(A_2) \neq \emptyset \ \forall A_1 \subseteq \Theta_1, A_2 \subseteq \Theta_2$, $A_1, A_2 \neq \emptyset$ (ρ_i denotes as usual the refining from Θ_i to $\Theta_1 \otimes \Theta_2$). Suppose by absurd that their common coarsening has more than a single element, $\Theta_1 \oplus \Theta_2 = \{a, b\}$. But then $\rho_1(\rho^1(a)) \cap \rho_2(\rho^2(b)) = \emptyset$, where ρ^i denotes the refining between $\Theta_1 \oplus \Theta_2$ and Θ_i, which goes against the hypothesis.
Induction step. Suppose that the thesis holds for $n-1$. Then, since $\Theta_1, ..., \Theta_n$ $\mathcal{IF}$ implies $\{\Theta_i, i \neq j\}$ $\mathcal{IF}$ $\forall j$, this implies by inductive hypothesis that

$$\bigoplus_{i \neq j} \Theta_i = \mathbf{0}_{\mathcal{F}} \quad \forall j = 1, ..., n.$$

Of course then, as $\mathbf{0}_{\mathcal{F}}$ is a coarsening of Θ_j $\forall j = 1, ..., n$,

$$\Theta_j \oplus \bigoplus_{i \neq j} \Theta_i = \Theta_j \oplus \mathbf{0}_{\mathcal{F}} = \mathbf{0}_{\mathcal{F}}.$$

We can use Lemma 1 to state that evidential independence of frames is indeed *stronger* than lattice-theoretic independence of frames in its first form.

Theorem 2. *$\Theta_1, ..., \Theta_n$ $\mathcal{IF}$ and $\Theta_j \neq \mathbf{0}_{\mathcal{F}}$ $\forall j$ then $\Theta_1, ..., \Theta_n$ $\mathcal{I}_1^*$.*

Proof. Let us suppose that $\Theta_1, ..., \Theta_n$ are $\mathcal{IF}$ but not $\mathcal{I}_1^*$, i.e. $\exists j : \Theta_j$ coarsening of $\bigotimes_{i \neq j} \Theta_i$.
We need to prove that $\exists A_1 \subseteq \Theta_1, ..., A_n \subseteq \Theta_n$ s.t. $\rho_1(A_1) \cap \cdots \cap \rho_n(A_n) = \emptyset$ where ρ_i denotes the refining from Θ_i to $\Theta_1 \otimes \cdots \otimes \Theta_n$ (Equation 2).
Since Θ_j is a coarsening of $\bigotimes_{i \neq j} \Theta_i$ then there exists a partition Π_j of $\bigotimes_{i \neq j} \Theta_i$ associated with Θ_j, and a refining ρ from Θ_j to $\bigotimes_{i \neq j} \Theta_i$.
As $\{\Theta_i, i \neq j\}$ are $\mathcal{IF}$, for all $\theta \in \bigotimes_{i \neq j} \Theta_i$ there exists $\theta_i \in \Theta_i$, $i \neq j$ s.t.

$$\{\theta\} = \bigcap_{i \neq j} \rho_i'(\Theta_i),$$

where ρ_i' is the refining to $\bigotimes_{i \neq j} \Theta_i$. Now, θ belongs to a certain element A of the partition Π_j. By hypothesis ($\Theta_j \neq \mathbf{0}_{\mathcal{F}}$ $\forall j$) Π_j contains at least two elements. But then we can choose $\theta_j = \rho^{-1}(B)$ with B another element of Π_j. In that case we obviously get

$$\rho_j(\theta_j) \cap \bigcap_{i \neq j} \rho_i(\theta_i) = \emptyset$$

which implies that $\{\Theta_i, i = 1, ..., n\} \neg\mathcal{IF}$ against the hypothesis.

However, the two notions are not equivalent: $\Theta_1, ..., \Theta_n \ \mathcal{I}_1^* \nvdash \Theta_1, ..., \Theta_n \ \mathcal{IF}$. Consider as a counterexample two frames Θ_1 and Θ_2 in which Θ_1 is not a coarsening of Θ_2 (Θ_1, Θ_2 are $\mathcal{I}_1^*$). Then $\Theta_1, \Theta_2 \neq \Theta_1 \otimes \Theta_2$ but it easy to find a situation (see Figure 6-left) in which Θ_1, Θ_2 are not $\mathcal{IF}$.

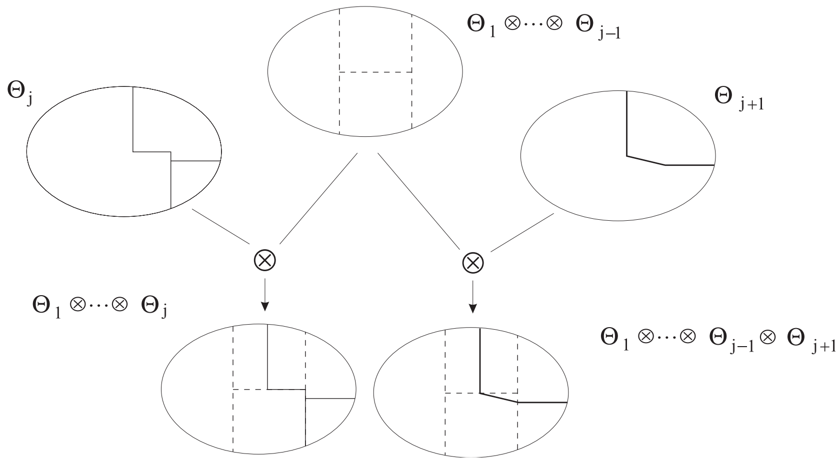

Fig. 5. A counterexample to $\mathcal{I}_2^* \vdash \mathcal{I}_1^*$.

More, it is easy to prove that $\mathcal{IF}$ is also *stronger than the second form* of lattice-theoretic independence.

Theorem 3. $\Theta_1, ..., \Theta_n \ \mathcal{IF} \vdash \Theta_1, ..., \Theta_n \ \mathcal{I}_2^*$.

Proof. We first need to show that $\Theta_1, ..., \Theta_n$ are $\mathcal{IF}$ iff the pair $\{\Theta_j, \otimes_{i\neq j}\Theta_i\}$ is $\mathcal{IF}$. As a matter of fact (3) can be written as

$$\Theta_j \otimes \bigotimes_{i\neq j} \Theta_i = \Theta_j \times (\times_{i\neq j}\Theta_i) \equiv \Big\{\Theta_j, \bigotimes_{i\neq j} \Theta_i\Big\} \ \mathcal{IF}.$$

But then by Lemma 1 we get as desired.

These two form of independence $\mathcal{I}_1^*, \mathcal{I}_2^*$ are not trivially related to each other: for instance, $\Theta_1, ..., \Theta_n \ \mathcal{I}_2^*$ does not imply $\Theta_1, ..., \Theta_n \ \mathcal{I}_1^*$. Figure 5 shows a counterexample: Given $\Theta_1 \otimes \cdots \otimes \Theta_{j-1}$ and Θ_j, one choice of Θ_{j+1} s.t. $\Theta_1, ..., \Theta_{j+1}$ are $\mathcal{I}_2^*$ but not $\mathcal{I}_1^*$ is shown.
It follows from Theorems 2 and 3 that, unless some frame is unitary, $\mathcal{IF} \vdash \mathcal{I}_1^* \wedge \mathcal{I}_2^*$. The converse is however false. Think of a pair of frames ($n = 2$), for which $\Theta_1 \oplus \Theta_2 \neq \Theta_1, \Theta_2$ ($\mathcal{I}_1^*$), $\Theta_1 \oplus \Theta_2 = \mathbf{0}_{\mathcal{F}}$ ($\mathcal{I}_2^*$). Now, those conditions are met by the counterexample of Figure 5, in which the two frames are not $\mathcal{IF}$.

4.4 Evidential independence is opposed to $\mathcal{I}_3^*$

On its side, lattice independence in its third form $\mathcal{I}_3^*$ is actually *incompatible with evidential independence.*

Theorem 4. *If $\Theta_1, ..., \Theta_n$ $\mathcal{IF}$, $n > 2$ then $\Theta_1, ..., \Theta_n \neg\mathcal{I}_3^*$. If Θ_1, Θ_2 $\mathcal{IF}$ then Θ_1, Θ_2 $\mathcal{I}_3^*$ iff $\exists \Theta_i = \mathbf{0}_\mathcal{F}$ $i \in \{1, 2\}$.*

Proof. According to (3), $\Theta_1, ..., \Theta_n$ are $\mathcal{IF}$ iff $|\otimes\Theta_i| = \prod_i |\Theta_i|$, while according to (8) they are $\mathcal{I}_3$ iff $|\Theta_1 \otimes \cdots \otimes \Theta_n| - 1 = \sum_i (|\Theta_i| - 1)$. They are both met iff

$$\sum_i |\Theta_i| - \prod_i |\Theta_i| = n - 1,$$

which happens only if $n = 2$ and either $\Theta_1 = \mathbf{0}_\mathcal{F}$ or $\Theta_1 = \mathbf{0}_\mathcal{F}$.

Stronger results hold when considering only pairs of frames. For $n = 2$ the relations (8) read as

$$\Theta_1 \oplus \Theta_2 \neq \Theta_1, \Theta_2, \qquad \Theta_1 \oplus \Theta_2 = \mathbf{0}_\mathcal{F}, \qquad |\Theta_1 \otimes \Theta_2| = |\Theta_1| + |\Theta_2| - 1. \quad (9)$$

Theorem 5. *If $\Theta_1, \Theta_2 \neq \mathbf{0}_\mathcal{F}$ then Θ_1, Θ_2 $\mathcal{I}_2^*$ implies Θ_1, Θ_2 $\mathcal{I}_1^*$.*
If $\exists \Theta_j = \mathbf{0}_\mathcal{F}$ $j \in \{1, 2\}$ then Θ_1, Θ_2 $\mathcal{I}_2^*$, $\mathcal{I}_3^*$, $\mathcal{IF}$, $\neg\mathcal{I}_1^*$.

Proof. The first fact is obvious from (9). If instead $\Theta_2 = \mathbf{0}_\mathcal{F}$ then by (9) $\Theta_1 \oplus \mathbf{0}_\mathcal{F} = \mathbf{0}_\mathcal{F} = \Theta_2$ and Θ_1, Θ_2 are not $\mathcal{I}_1^*$ while they are $\mathcal{I}_2^*$. As $|\Theta_1 \otimes \Theta_2| = |\mathbf{0}_\mathcal{F}| \cdot |\Theta_1| = |\Theta_1|$, $|\Theta_2| = 1$ in that case Θ_1, Θ_2 are $\mathcal{I}_3^*$ again by (9). Finally, according to (3), they are $\mathcal{IF}$ as $|\Theta_1 \otimes \Theta_2| = |\Theta_1| = 1 \cdot |\Theta_1| = |\Theta_2||\Theta_1|$.

For the binary partitions of Θ, i.e. the *atoms* (elements covering $\mathbf{0}$) A^* of the lattice $L^*(\Theta)$, Theorem 4 implies that $\mathcal{IF}$ and $\mathcal{I}_3^*$ are incompatible.

Corollary 1. *If $\Theta_1, ..., \Theta_n \in A^*$ then $\Theta_1, ..., \Theta_n$ $\mathcal{IF}$ implies $\Theta_1, ..., \Theta_n$ $\neg\mathcal{I}_3^*$.*

On the other side, the other two relations are trivial for atoms of $L^*(\Theta)$.

Theorem 6. *If $\Theta_1, ..., \Theta_n \in A^*$ then $\Theta_1, ..., \Theta_n$ are both $\mathcal{I}_1^*$ and $\mathcal{I}_2^*$.*

Proof. If $\Theta_j \in A^*$ $\forall j$ then $\Theta_j \oplus \bigotimes_{i \neq j} \Theta_i = \mathbf{0}_\mathcal{F} \neq \Theta_j$ $\forall j$ and $\Theta_1, ..., \Theta_n$ are $\mathcal{I}_1^*$. But then by Equation 8 $\Theta_1, ..., \Theta_n$ are also $\mathcal{I}_2^*$.

As a matter of fact, evidential independence and $\mathcal{I}_3^*$ are in opposition for pairs of atoms of $L^*(\Theta)$.

Theorem 7. *$\Theta_1, \Theta_2 \in A^*$ are $\mathcal{IF}$ iff $\Theta_1, \Theta_2 \neg\mathcal{I}_3^*$.*

Proof. By Theorem 4 we have that $\mathcal{IF} \vdash \neg\mathcal{I}_3^*$. To prove the reverse implication $\mathcal{I}_3^* \vdash \neg\mathcal{IF}$ we just need to notice that $\Theta_1, \Theta_2 \in A^*$ are $\mathcal{I}_3^*$ iff $|\Theta_1 \otimes \Theta_2| = |\Theta_1| + |\Theta_2| - 1 = 2 + 2 - 1 = 3$ while for them to be $\mathcal{IF}$ it has to be $|\Theta_1 \otimes \Theta_2| = |\Theta_1||\Theta_2| = 2 \cdot 2 = 4$.

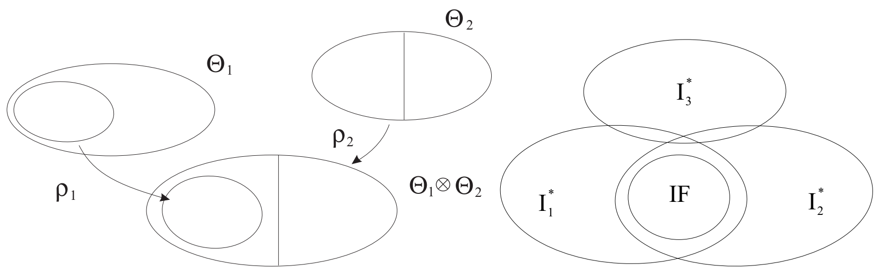

Fig. 6. Left: A counterexample to $\mathcal{I}_1^* \vdash \mathcal{IF}$. Right: Relations between independence of frames $\mathcal{IF}$ and all different forms of semi-modular independence on the lower semi-modular lattice of frames $L^*(\Theta)$.

5 Comments and conclusions

Independence of sources in the theory of evidence can be reduced to independence of frames. This shows in turn intriguing formal analogies with linear independence. In this paper we proved that families of frames share indeed with projective geometries the algebraic structure of semi-modular lattice (Theorem 1). Several forms of independence relations can be introduced on the elements of such lattices, and related with Shafer's independence of frames.

Figure 6-right illustrates what we have learned about how $\mathcal{IF}$ relates to the various forms of lattice-theoretic independence in the lower semi-modular lattice of frames, in the general case of a collection of more than two non-atomic frames (the case $\Theta_i = \mathbf{0}_\mathcal{F}$ is neglected). Evidential independence appears distinct from but related to lattice-theoretic independence.
This is even more interesting when we consider that condition (2) comes directly from the notion of independence of frames as Boolean sub-algebras [18]. Boolean independence $\mathcal{IF}$ is *a stronger condition* than both $\mathcal{I}_1^*$ and $\mathcal{I}_2^*$ (Theorems 2, 3) which are indeed trivial for binary partitions of Θ (Theorem 6). On the other side $\mathcal{IF}$ and $\mathcal{I}_3^*$ are mutually exclusive (Theorems 4 and 7, Corollary 1). As $\mathcal{I}_3^*$ is in turn a form of matroidal independence [13] this sheds new light on the relation between Boolean algebra and matroid theory.

The prosecution of this study, particularly in the context of matroid theory, could in the future shed some more light on both the nature of independence of sources in the theory of subjective probability, and the relationship between lattice, matroidal and Boolean independence in discrete mathematics, pointing out the necessity of a more general, comprehensive definition of this very useful and widespread notion.

References

1. F. Campos and F.M.C. de Souza, *Extending Dempster-Shafer theory to overcome counter intuitive results*, Proceedings of IEEE NLP-KE '05, vol. 3, 2005,

pp. 729– 734.
2. J. Carlson and R.R. Murphy, *Use of Dempster-Shafer conflict metric to adapt sensor allocation to unknown environments*, Tech. report, Safety Security Rescue Research Center, University of South Florida, 2005.
3. M. E. G. V. Cattaneo, *Combining belief functions issued from dependent sources.*, ISIPTA, 2003, pp. 133–147.
4. F. Cuzzolin, *Algebraic structure of the families of compatible frames of discernment*, AMAI **45(1-2)** (2005), 241–274.
5. A.P. Dempster, *Upper and lower probabilities generated by a random closed interval*, Annals of Mathematical Statistics **39** (1968), 957–966.
6. M. Deutsch-McLeish, *A study of probabilities and belief functions under conflicting evidence: comparisons and new method*, Proceedings of IPMU'90, Paris, France, 2-6 July 1990, pp. 41–49.
7. D. Dubois and H. Prade, *On the combination of evidence in various mathematical frameworks*, Reliability Data Collection and Analysis (J. flamm and T. Luisi, eds.), 1992, pp. 213–241.
8. A. Josang, M. Daniel, and P. Vannoorenberghe, *Strategies for combining conflicting dogmatic beliefs*, Proceedings of Fusion 2003, vol. 2, 2003, pp. 1133–1140.
9. Jurg Kohlas and Paul-André Monney, *A mathematical theory of hints - an approach to the dempster-shafer theory of evidence*, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, 1995.
10. E. Lefevre, O. Colot, and P. Vannoorenberghe, *Belief functions combination and conflict management*, Information Fusion Journal **3** (2002), no. 2, 149–162.
11. W. Liu, *Analyzing the degree of conflict among belief functions*, Artif. Intell. **170** (2006), no. 11, 909–924.
12. C.K. Murphy, *Combining belief functions when evidence conflicts*, Decision Support Systems **29** (2000), 1–9.
13. J. G. Oxley, *Matroid theory*, Oxford University Press, Great Clarendon Street, Oxford, UK, 1992.
14. K. Sentz and S. Ferson, *Combination of evidence in Dempster-Shafer theory*, Tech. report, SANDIA Tech. Report, SAND2002-0835, April 2002.
15. G. Shafer, *A mathematical theory of evidence*, Princeton University Press, 1976.
16. ———, *Perspectives on the theory and practice of belief functions*, International Journal of Approximate Reasoning **4** (1990), 323–362.
17. G. Shafer, P.P. Shenoy, and K. Mellouli, *Propagating belief functions in qualitative Markov trees*, IJAR **1** (1987), (4), 349–400.
18. R. Sikorski, *Boolean algebras*, Springer Verlag, 1964.
19. P. Smets, *The combination of evidence in the transferable belief models*, IEEE Transactions on PAMI **12** (1990), 447–458.
20. ———, *Upper and lower probability functions versus belief functions*, Proceedings of the International Symposium on Fuzzy Systems and Knowledge Engineering, Guangzhou, China, 1987, pp. 17–21.
21. G. Szasz, *Introduction to lattice theory*, Academic Press, New York, 1963.
22. M.J. Wierman, *Measuring conflict in evidence theory*, Proceedings of the Joint 9th IFSA World Congress, Vancouver, BC, Canada, vol. 3, 2001, pp. 1741–1745.
23. R.R. Yager, *On the Dempster-Shafer framework and new combination rules*, Information Sciences **41** (1987), 93–138.
24. L. Zadeh, *A simple view of the Dempster-Shafer theory of evidence and its implication for the rule of combination*, AI Magazine **7** (1986), no. 2, 85–90.