Simplicial complexes of finite fuzzy sets

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Abstract

In this paper we extend our geometric approach to the theory of evidence in order to include other important classes of finite fuzzy measures. In particular we describe the geometric counterparts of possibility measures or fuzzy sets, represented as consonant belief functions. The correspondence between chains of subsets and convex sets of consonant functions is studied and its properties analyzed, eventually yielding an elegant representation of the region of consonant belief functions in terms of the notion of simplicial complex.

Keywords: Theory of evidence, belief space, fuzzy and possibility measures, consonant belief functions, simplicial complex.

1 Introduction

Uncertainty measures are assuming a mayor role in fields like artificial intelligence, where problems involving formalized reasoning or machine learning are common. The theory of evidence (ToE) [13] is one the most popular approaches, being quite a natural extension of the classical Bayesian formalism. In the ToE probabilities are replaced by *belief functions* (b.f.), which assign values between 0 and 1 to subsets of the sample space instead of single elements. Bayes' rule is also replaced by a more general operation called *Dempster's sum* [4] which rules the combination of two or more belief functions. *Possibility measures* can also be seen as peculiar b.f.s, namely the so called *consonant* belief functions. As it is been already noticed, this implies a connection between fuzzy theory and theory of evidence.

In this paper we propose a geometric picture of the connections between those varieties of fuzzy measures based on the geometric approach to the theory of evidence developed in the last two years. In this framework, belief functions are represented by points of a convex space called *belief space* [3]. We show that consonant b.f.s are in correspondence with chains of subsets of their domain, and are hence located in a collection of connected regions of the belief space assuming the form of a *simplicial complex.* We also discuss a possible interpretation of these results, in particular the existence of a duality between probabilities and possibilities due to their close relation to the L_1 and L_∞ norms respectively.

1.1 Previous work

The geometric approach to the theory of evidence and generalized probabilities is due to the author, but a close reference is perhaps a recent paper of Ha and Haddawy [9] where they exploit methods of convex geometry to represent probability intervals. P. Black's interesting results on the geometry of belief functions and other monotone capacities can instead be found in [5], where he uses shapes of geometric loci to give a direct visualization of distinct classes of monotone capacities.

Many authors, like Yager [15] and Romer [12] among the others, have on the other side studied the connection between fuzzy numbers and Dempster-Shafer theory. For instance, Klir *et al.* published an excellent discussion [11] on the relations among fuzzy and belief measures and possibility theory. The material exposed in Section 4 is largely abstracted from this paper. S. Heilpern [10] also presented the theoretical background of fuzzy numbers connected with the possibility and Dempster-Shafer theories, describing some types of representation of fuzzy numbers and studying the notions of distance and order between fuzzy numbers based on these representations. Caro and Nadjar [1], instead, suggested a generalization of the Dempster-Shafer theory to a fuzzy valued measure.

The points of contact between evidential formalism (in the transferable belief model implementation) and possibility theory has been briefly investigated by Ph. Smets in [14], while Dubois and Prade [7] has worked on the consonant approximation of belief functions.

2 The theory of evidence

Following Shafer [13] we call the finite set of possible outcomes of a decision problem *frame of discernment*.

Definition 1. *A* basic probability assignment (b.p.a.) *over a frame* Θ *is a function* $m : 2^\Theta \rightarrow [0,1]$ *such that*

$$m(\emptyset) = 0, \quad \sum_{A \subset \Theta} m(A) = 1, \quad m(A) \geq 0 \; \forall A.$$

Subsets of Θ associated with non-zero values of m are called *focal elements* and their union *core*.

Definition 2. *The* belief function $s : 2^\Theta \rightarrow [0,1]$ *associated with the basic probability assignment* m *is defined as:*

$$s(A) = \sum_{B \subset A} m(B).$$

Conversely, the basic probability assignment m associated with a given belief function s can be uniquely recovered by means of the *Moebius inversion formula*

$$m(A) = \sum_{B \subset A} (-1)^{|A-B|} s(B) \qquad (1)$$

so that there is a 1-1 correspondence between the two set functions $m \leftrightarrow s$.

An alternative mathematical representation of the evidence encoded by belief function s is the *upper probability* function $P^*_s : 2^\Theta \rightarrow [0,1]$,

$$P^*_s(A) \doteq 1 - s(A^c) = 1 - \sum_{B \subset A^c} m(B) \qquad (2)$$

whose value $P^*_s(A)$ expresses the plausibility of a proposition A or, in other words, the amount of evidence *not against* A. Again, P^*_s convey the same information of s, and can be expressed as

$$P^*_s(A) = \sum_{B \cap A \neq \emptyset} m(B) \geq s(A).$$

In the simplest situation the evidence points *to a single non-empty subset* $A \subset \Theta$.

Definition 3. *A belief function* $s : 2^\Theta \rightarrow [0,1]$ *is called* simple support function *focused on* A *if its b.p.a. is given by* $m(A) = \sigma$, $m(\Theta) = 1 - \sigma$ *and* $m(B) = 0$ *for every other* B*, where* $0 \leq \sigma \leq 1$.

However, belief functions can support more than one proposition at a time. In particular, in the theory of evidence a probability function is simply a peculiar belief function which satisfies the additivity rule for disjoint sets (*Bayesian* b.f.). It can be proved that s is Bayesian iff

$$m(A) = 0, |A| > 1.$$

At the opposite of Bayesian functions stand the so-called *consonant* belief functions.

Definition 4. *A belief function is said to be consonant if its focal elements are nested.*

Proposition 1 [13] illustrates some of their properties.

Proposition 1. *If s is a belief function with upper probability function P^*, then the following conditions are equivalent:*

1. *s is consonant;*
2. *$s(A \cap B) = \min(s(A), s(B))$ for every $A, B \subset \Theta$;*
3. *$P_s^*(A \cup B) = \max(P_s^*(A), P_s^*(B))$ for every $A, B \subset \Theta$;*
4. *$P_s^*(A) = \max_{\theta \in A} P_s^*(\{\theta\})$ for all non-empty $A \subset \Theta$.*

3 Belief space

Motivated by the search for an adequate probabilistic approximation of belief functions, in some previous works of ours we introduced the notion of *belief space* [3], as the space of all the belief functions we can define on a given domain.

Consider a frame of discernment Θ and introduce in the Euclidean space $\mathbb{R}^{|2^\Theta|-1}$ an orthonormal reference frame $\{X_A\}_{A \subset \Theta}$ in which each coordinate function x_A measures the belief value associated with the subset A of Θ.

Definition 5. *The* belief space *associated with Θ is the set of points $\mathcal{S}$ of $\mathbb{R}^{|2^\Theta|-1}$ corresponding to a belief function.*

It is not difficult to prove by means of the positivity axiom of belief functions (see [3] for the details) that $\mathcal{S}$ is convex. After denoting with P_A the unique belief function assigning all the mass to a single subset A of Θ,

$$m_s(A) = 1, \quad m_s(B) = 0 \; \forall \; B \neq A$$

we can give an exact expression for the belief space. Here m_s is the b.p.a. associated with s. It can be proved that (see [3] again), calling $\mathcal{E}_s$ the list of focal elements of s,

Theorem 1. *The set of all the belief functions with focal elements in a given collection $\mathcal{F}$ is closed and convex in $\mathcal{S}$:*

$$\{s : \mathcal{E}_s \subset \mathcal{F}\} = Cl(\{P_A : A \in \mathcal{F}\}).$$

The simplicial form of the belief space is then just a trivial consequence of Theorem 1.

Corollary 1. *The belief space $\mathcal{S}$ coincides with the convex closure of all the basic belief functions P_A,*

$$\mathcal{S} = Cl(P_A, \; A \subset \Theta, \; A \neq \emptyset). \tag{3}$$

Notice that the vectors $\{P_A, A \subset \Theta\}$ are *linearly independent* [2] so that the dimension of $\mathcal{S}$ is $2^{|\Theta|} - 2$. Furthermore, any belief function $s \in \mathcal{S}$ can be written as a convex sum as follows:

$$s = \sum_{A \subset \Theta, \; A \neq \emptyset} m_s(A) \cdot P_A. \tag{4}$$

Figure 1 illustrates the concept.

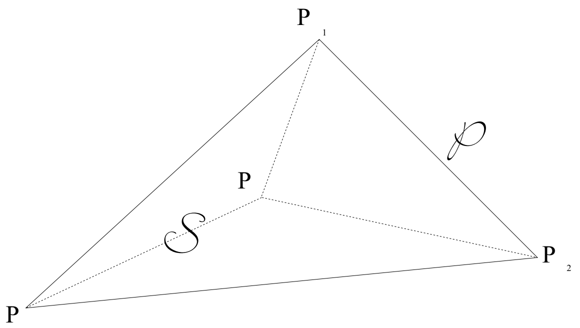

Figure 1: simplicial structure of the belief space. Its vertices are all the basic belief functions P_A. The probabilistic subspace is a subset $Cl(P_{\{\theta_i\}}, \; i = 1, .., |\Theta|)$ of its border.

Clearly, since a probability is a belief function assigning non zero masses to singletons only, Theorem 1 yields the following

Corollary 2. *The set $\mathcal{P}$ of all the Bayesian belief functions is a subset of the border of $\mathcal{S}$, precisely the simplex determined by all the basis functions associated with singletons:*

$$\mathcal{P} = Cl(P_{\{\theta_i\}}, \; i = 1, ..., |\Theta|).$$

3.1 Binary frame

As an example let us consider a frame of discernment containing only two elements, $\Theta = \{x, y\}$. The non-empty subsets of Θ are Θ itself, $\{x\}$ and $\{y\}$: hence the belief space will be the 2-dimensional simplex $Cl(P_\Theta, P_x, P_y)$ ($2^{|\Theta|} - 2 = 2$). Being $s(\Theta) = 1$ for each belief function, the Θ-coordinate can be neglected and we can write

$$P_\Theta = (0,0), \quad P_x = (1,0), \quad P_y = (0,1).$$

The belief space can then be represented as a triangle on a plane (Figure 2).

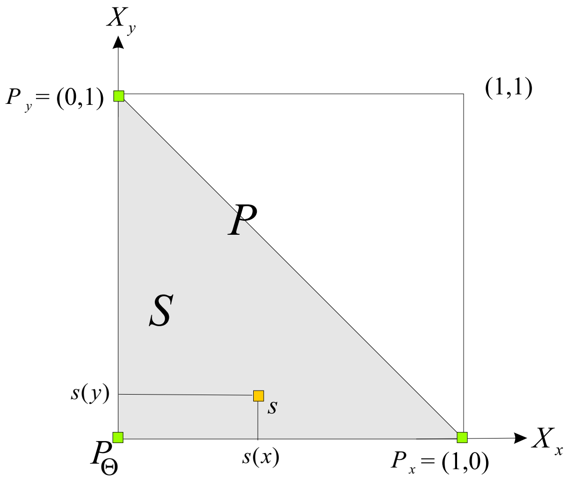

Figure 2: geometry of the binary belief space.

$\mathcal{P}$ is in this case the diagonal line segment $Cl(P_x, P_y)$. Each belief function $s \in \mathcal{S}$ has coordinates $(s(x), s(y)) = (m_s(x), m_s(y))$.

4 Fuzzy measures

4.1 Belief functions as fuzzy measures

Evidential reasoning and related theories are often confused with fuzzy theory. In fact, fuzzy measures are a generalization of belief measures.

Definition 6. *Given a frame Θ and a non-empty family $\mathcal{F}$ of subsets of Θ, a* fuzzy measure *μ on $\langle\Theta, \mathcal{F}\rangle$ is a function*

$$\mu : \mathcal{F} \to [0,1]$$

satisfying the following conditions:

1. $\mu(\emptyset) = 0$;
2. *if $A \subseteq B$ then $\mu(A) \leq \mu(B)$, for every $A, B \in \mathcal{F}$;*
3. *for any increasing sequence $A_1 \subseteq A_2 \subseteq \cdots$ of subsets in $\mathcal{F}$,*

   $$\text{if } \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}, \text{ then } \lim_{i\to\infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i)$$

   (continuity from below)*;*
4. *for any decreasing sequence $A_1 \supseteq A_2 \supseteq \cdots$ of subsets in $\mathcal{F}$,*

   $$\text{if } \bigcap_{i=1}^{\infty} A_i \in \mathcal{F} \text{ and } \mu(A_1) < \infty,$$

   $$\text{then } \lim_{i\to\infty} \mu(A_i) = \mu(\bigcap_{i=1}^{\infty} A_i)$$

   (continuity from above).

It is easy to see from the above definition that

Proposition 2. *Belief measures are fuzzy measures.*

4.2 Possibilities and fuzzy sets

Besides the evidential reasoning, another important uncertainty theory, called *possibility theory* [6] is based on a special fuzzy measure called *possibility measure*.

Definition 7. *A* possibility measure *on a domain Θ is a function*

$$Pos : 2^\Theta \to [0,1]$$

such that $Pos(\emptyset) = 0$, $Pos(\Theta) = 1$ and

$$Pos(\bigcup_i A_i) = \sup_i Pos(A_i)$$

for any family $\{A_i | A_i \in 2^\Theta, i \in I\}$ where I is an arbitrary set index.

Any possibility measure is uniquely characterized by a *membership function*

$$\begin{array}{rccc} \pi : & \Theta & \to & [0,1] \\ & x & \mapsto & \pi(x) \doteq Pos(\{x\}) \end{array}$$

via the formula $Pos(A) = \sup_{x\in A} \pi(x)$.

It just needs to look at condition 4 of Proposition 1 to realize that, for consonant belief functions, the restriction of the plausibility function to singletons $P_s^*(\{x\})$ plays the role of the membership function $\pi(x)$ for a possibility measure.
In other words,

Proposition 3. *The upper probability function P_s^* associated with a belief function s on a domain Θ is a possibility measure iff s is a consonant belief function.*

Hence possibility theory is embedded into the ToE, where possibility measures are represented by consonant belief functions.

5 Consonant subspace

The geometric interpretation of belief functions puts the results of Section 4 in a different light. Using the convex geometry of the belief space we can pose the problem of finding the region of $\mathcal{S}$ whose points correspond to consonant belief functions (and therefore to fuzzy sets, Section 4.2).

5.1 Chains of subsets as consonant belief functions

The most natural thing to do is to observe that, where generic belief functions do not undergo to restrictions on their list of focal elements, consonant belief functions are characterized by the fact that their focal elements can be rearranged into an ordered list.

The power set 2^Θ of a frame is a *partially ordered set* with respect to the set-theoretic inclusion. In other words, the relation $\subset$ possess three properties: reflexivity ($A \subset A$ $\forall A \in 2^\Theta$), antisymmetry ($A \subset B$ and $B \subset A$ implies $A = B$), and transitivity ($A \subset B$ and $B \subset C$ implies $A \subset C$). A *chain* of a poset is a collection of pairwise comparable elements (*totally ordered set*).
The possible lists of focal elements associated with consonant belief functions then correspond to all the possible chains of subsets

$$A_1 \subset ... \subset A_m$$

in the partially ordered set $(2^\Theta, \subset)$.

Now, Theorem 1 implies that all the b.f.s whose focal elements belong to a chain $X = \{A_1, ..., A_m\}$ is $Cl(P_{A_1}, ..., P_{A_m})$. No matter what the basic probability assignment is, all the $s \in Cl(P_{A_1}, ..., P_{A_m})$ are consonant belief functions.

Let us denote with $n \doteq |\Theta|$ the cardinality of the frame Θ. Clearly, since each chain in $(2^\Theta, \subset)$ is a subset of a maximal chain (a chain including subsets of any size from 1 to n), the region of consonant belief functions turns out to be the union of a collection of convex components, each associated with a maximal chain $\mathcal{A}$:

$$\mathcal{C} = \bigcup_{\mathcal{A}=A_1 \subset ... \subset A_n} Cl(P_{A_1}, ..., P_{A_n}).$$

The number of convex components of $\mathcal{C}$ is then the number of maximal chains in $(2^\Theta, \subset)$, i.e.

$$\prod_{k=1}^{n} \binom{k}{1} = n!$$

since given a size k set we can build a new set containing it by just choosing one of the remaining elements. Since the length of a maximal chain is again the cardinality of Θ, the dimension of these convex components is $\dim Cl(P_{A_1}, ..., P_{A_n}) = n - 1$.

Each basic belief function P_B obviously belongs to several distinct components. In particular, if $|B| = k$ the total number of maximal chains containing B is

$$(n-k)!k! \qquad (5)$$

since in the power set of B the number of maximal chains is $k!$, while to get a chain from B to Θ we just have to add an element of B^c (whose size is $n - k$) at each step. (5) is also the number of convex components of $\mathcal{C}$ containing P_B.

In particular, each vertex $P_{\{x\}}$ of the probabilistic subspace $\mathcal{P}$ (for which $|\{x\}| = k = 1$) belongs to a sheaf of $(n-1)!$ convex components of the consonant subspace. Clearly the maximum number of simplices is $n!$, obtained for $k = n$ (the vacuous belief function P_Θ).

An obvious remark is that $\mathcal{C}$ *is connected*, for each convex component is obviously connected, and each pair of convex components of the consonant subspace has at least P_Θ as intersection.

5.2 Ternary case

Let us consider, as an example, the case of a frame of size 3: $\Theta = \{x, y, z\}$. The maximal chains are then

$$\begin{array}{cc} \{x\} \subset \{x,z\} \subset \Theta & \{y\} \subset \{x,y\} \subset \Theta \\ \{x\} \subset \{x,y\} \subset \Theta & \{y\} \subset \{y,z\} \subset \Theta \end{array}$$
$$\{z\} \subset \{y,z\} \subset \Theta$$
$$\{z\} \subset \{x,z\} \subset \Theta$$

Each singleton is then associated with 2 chains, and the total number of convex components, whose dimension is $|\Theta| - 1 = 2$,

$$\begin{array}{cc} Cl(P_{\{x\}}, P_{\{x,z\}}, P_\Theta) & Cl(P_{\{y\}}, P_{\{x,y\}}, P_\Theta) \\ Cl(P_{\{x\}}, P_{\{x,y\}}, P_\Theta) & Cl(P_{\{y\}}, P_{\{y,z\}}, P_\Theta) \\ \multicolumn{2}{c}{Cl(P_{\{z\}}, P_{\{y,z\}}, P_\Theta)} \\ \multicolumn{2}{c}{Cl(P_{\{z\}}, P_{\{x,z\}}, P_\Theta)} \end{array}$$

is $3! = 6$.

The reader can realize how each 2-dimensional convex component (for instance $Cl(P_{\{x\}}, P_{\{x,z\}}, P_\Theta)$) has an intersection of dimension $|\Theta| - 2 = 1$ ($Cl(P_{\{x,z\}}, P_\Theta)$) with a single other component ($Cl(P_{\{z\}}, P_{\{x,z\}}, P_\Theta)$) associated with a different element of Θ.

The geometry of the ternary frame can then be represented as in Figure 3, where the belief space is 6-dimensional $\mathcal{S}_3 = Cl(P_{\{x\}}, P_{\{y\}}, P_{\{z\}}, P_{\{x,y\}}, P_{\{x,z\}}, P_{\{y,z\}}, P_\Theta)$, its probabilistic subspace is a 2-dimensional simplex $\mathcal{P}_3 = Cl(P_{\{x\}}, P_{\{y\}}, P_{\{z\}})$, and the consonant subspace $\mathcal{C}_3$ is given by the union of the connected components listed above.

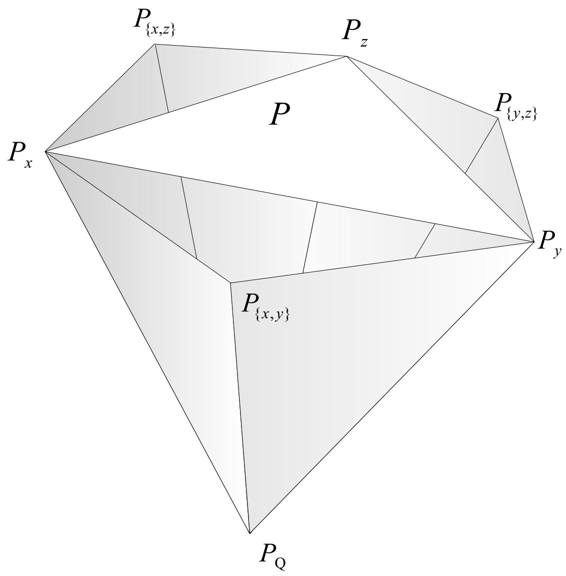

Figure 3: pictorial representation of the simplicial complex of the consonant belief functions for a ternary frame Θ_3. The complex is composed by $n! = 3! = 6$ convex components of dimension $n - 1 = 2$, each vertex of $\mathcal{P}_3$ being shared by $(n-1)! = 2! = 2$ of them. The region is connected, and is part of the border $\partial\mathcal{S}_3$ of the belief space $\mathcal{S}_3$.

5.3 Consonant subspace as simplicial complex

These properties of $\mathcal{C}$ can be summarized by means of another concept of convex geometry, slightly more general than that of simplex [8].

Definition 8. *An n-dimensional simplex is the convex closure of $n+1$ points of the Euclidean space, $\sigma = [\alpha_0, ..., \alpha_n]$.*

The *faces* of an n-dimensional simplex are all the possible simplices generated by a subset of its vertices, i.e. $[\alpha_{j_1}, ..., \alpha_{j_k}]$ with $\{j_1, ..., j_k\} \subset \{1, ..., n\}$. The $n-1$ dimensional faces are obtained by simply eliminating one vertex. The i-th face of $\sigma = [\alpha_0, ..., \alpha_n]$ is denoted by

$$\sigma_i^{n-1} = [\alpha_0, ..., \hat{\alpha}_i, ..., \alpha_n].$$

Lower dimensional faces are obtained by erasing an arbitrary number of vertices.

Definition 9. *The oriented boundary of the simplex $\sigma^n = [\alpha_0, ..., \alpha_n]$ is a formal linear combination of its faces of the form*

$$\partial\sigma^n = \sum_{i=0}^{n} (-1)^i \sigma_i^{n-1}.$$

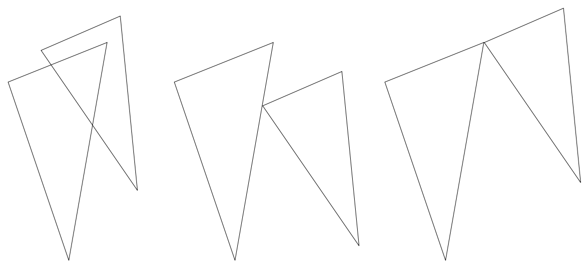

Figure 4: constraints on the intersection of simplices in a complex.

Definition 10. *A* simplicial complex *is a collection Σ of simplices of arbitrary dimension possessing the following properties:*

1. *if a simplex belongs to Σ, then all its faces of any dimension belong to Σ;*

2. *the intersection of two d-dimensional simplices is a face of both the intersecting simplices.*

Let us consider for instance two triangles on the plane (2-dimensional simplices). Roughly speaking, the second condition says that the intersection of those triangles cannot contain points of their interiors (Figure 4 left). It cannot also be any subset of their borders (middle), but has to be a face (right, in this case a single vertex). Note that if two simplices intersect in a face τ, they obviously intersect in every face of τ.

Theorem 2. *$\mathcal{C}$ is a simplicial complex included in the belief space $\mathcal{S}$.*

Proof. Property 1 of Definition 10 is trivially satisfied, since if a simplex $Cl(P_{A_1}, ..., P_{A_n})$ correspond to a chain $A_1 \subset ... \subset A_n$ in the poset $(2^\Theta, \subset)$, clearly any face of this simplex correspond to a subchain in 2^Θ, and then to a simplex of consonant belief functions.

About property 2, let us consider the intersection of two convex components

$$Cl(P_{A_1}, ..., P_{A_n}) \cap Cl(P_{B_1}, ..., P_{B_n})$$

associated with the pair of maximal chains $\mathcal{A} = A_1, ..., A_n$ and $\mathcal{B} = B_1, ..., B_n$ respectively (where $A_n = B_n = \Theta$). Being the vectors $\{P_A\}$ linearly independent, no linear combination of P_B's can yield an element of $span(P_{A_1}, ..., P_{A_n})$, unless some of those vectors coincide. In this case the desired intersection is

$$Cl(P_{C_{i_1}}, ..., P_{C_{i_k}})$$

where

$$C = \{C_{i_j}, j = 1, ..., k\} = \mathcal{A} \cap \mathcal{B} \qquad (6)$$

with $k < n$ and $C_{i_k} = \Theta$. But then C is a subchain of both $\mathcal{A}$ and $\mathcal{B}$, so that (6) is a face of both $Cl(P_{A_1}, ..., P_{A_n})$ and $Cl(P_{B_1}, ..., P_{B_n})$. $\square$

As Figure 3 shows, $\mathcal{P}$ and the components of $\mathcal{C}$ have the same dimension, and are both parts of the boundary $\partial\mathcal{S}$ of the belief space.

6 Discussion

The geometric description of consonant belief functions in the belief space clearly pictures a sort of duality between probability and possibility measures, represented by the dichotomy simplex - simplicial complex. It is not hard to show that this is due to the connection of those measures with the norms L_1 and L_∞ respectively, i.e. $P(A) = \sum_{x \in A} P(x)$, $Pos(A) = \max_{x \in A} Pos(x)$.

The well-known problem of finding a consonant approximation of a belief function faced by Dubois and Prade in [7] can then be approached from a geometric point of view too. In fact, we have recently proposed an approximation criterion based on Dempster's rule, in which the solution minimizes the integral difference between all the combinations of the original b.f. s and its approximation $\hat{s}$ with any other belief function t [16]

$$\hat{s} = \arg\min_{s' \in \mathcal{C}} \int_{t \in \mathcal{S}} dist(s \oplus t, s' \oplus t) dt \qquad (7)$$

where $t \in \mathcal{S}$ is an arbitrary belief function on the same frame, $dist$ is a distance function in the Euclidean space (being the belief space a subset of $\mathbb{R}^N$), and $\mathcal{C}$ is the class of belief functions the approximation belongs to.
For probabilistic approximations this yields a unique solution no matter what is the choice of the norm, namely the relative plausibility function $\tilde{P}_s^*$

$$\tilde{P}_s^*(A) = \frac{\sum_{\theta \in A} P_s^*(\{\theta\})}{\sum_{\theta \in \Theta} P_s^*(\{\theta\})} = \frac{\|v_A\|_1}{\|v_\Theta\|_1}$$

where v_A is the vector $[P_s^*(\theta_1), ..., P_s^*(\theta_{|A|})]$ of the plausibilities of the elements of A.

We are currently aiming at a general proof for the probabilistic case, based on the *representation* property: the relative plausibility of singletons $\tilde{P}_s^*$ is a perfect representation of s in the probability subspace through Dempster's rule, i.e. $s \oplus t = \tilde{P}_s^* \oplus t$, $\forall t \in \mathcal{P}$. The convex geometry of Dempster's rule [2] can then lead us to a geometric solution of the approximation problem (7).

The duality principle would imply to choose as possibilistic approximation of a belief function s the unique consonant belief function c with plausibility

$$P_c^*(A) = \frac{\max_{\theta \in A} P_s^*(\{\theta\})}{\max_\theta P_s^*(\{\theta\})} = \frac{\|v_A\|_\infty}{\|v_\Theta\|_\infty}. \quad (8)$$

However, the behavior of (8) in the Dempster-based approximation problem and its characterization in terms of focal elements are still open problems, as the exciting option of exploiting results of the theory of chain and simplicial complexes to widen our knowledge of fuzzy measures.

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