

# The intersection probability and its properties

**Fabio Cuzzolin**

Department of Computing  
**Oxford Brookes University**

ECSQARU 2009 - Verona, Italy  
July 1 – 3 2009

# Outline

- 1 Approximation of probability intervals
- 2 Intersection probability
- 3 Credal geometry in the probability simplex
- 4 Affine family of transformations
- 5 Conclusions

# Uncertainty measures as credal sets

- assume the possible answers to a problem  $Q$  form a finite set  $\Theta = \{x_1, \dots, x_n\}$  (frame)
- given some evidence we can describe our belief on the outcome of  $Q$  in different ways
- random sets, belief functions, probability intervals
- correspond to **credal sets**, i.e., convex sets of probability distributions

# The probability transformation problem for BFs

- **probability transformation**: reducing a complex uncertainty measure to a single probability
- problem has been widely studied for BFs
- reasons: efficient evidence combination, decision making
- others have argued that Bayesian and belief calculi have the same expressive power
- proposals: pignistic function [Smets], relative plausibility [Voorbraak, Shenoy], relative belief [Cuzzolin], orthogonal projection [Cuzzolin], Sudano, etc

# Probability intervals

- **interval probability system**: system of constraints on the probability values of a distribution  $p : \Theta \rightarrow [0, 1]$  on  $\Theta$  of the form:

$$(l, u) \doteq \left\{ l(x) \leq p(x) \leq u(x), \forall x \in \Theta \right\}$$

- determines a convex (credal) set of probabilities
- example: probability interval on a domain  $\Theta = \{x, y, z\}$

$$0.2 \leq p(x) \leq 0.8, \quad 0.4 \leq p(y) \leq 1, \quad 0 \leq p(z) \leq 0.4.$$

# Goal: transformation of prob intervals

- goal: sensible **probability transformation of probability intervals**
- need for sensible rationality principle (as in the pignistic case)
- proposal: **intersection probability**, originally found in the geometric approach to BFs [Cuzzolin IEEE SMC-B'07]
- it turns out to be a transformation of prob intervals, obeying a simple rationality axiom

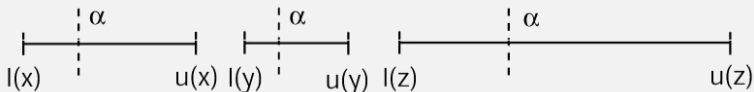
# Rationale

## of the intersection probability

- all the intervals  $[l(x), u(x)]$ ,  $x \in \Theta$  have the same relevance in a set of constraints
- reasonable request: the desired probability behave **homogeneously in each element**  $x$  of the frame  $\Theta$

$$p(x) = l(x) + \alpha(u(x) - l(x)) \quad \forall x \in \Theta$$

for some  $\alpha \in [0, 1]$



# Intersection probability

analytic form

- unique solution: **intersection probability**

$$p[(I, u)](x) = \beta[(I, u)] \cdot u(x) + (1 - \beta[(I, u)]) \cdot I(x)$$

where

$$\beta[(I, u)] = \frac{1 - \sum_{x \in \Theta} I(x)}{\sum_{x \in \Theta} (u(x) - I(x))}$$



## Relative uncertainty of singletons

- most interesting interpretation of  $p[(l, u)]$  comes from its alternative form

$$p[(l, u)](x) = l(x) + \left(1 - \sum_x l(x)\right) R[(l, u)](x)$$

where

$$R[(l, u)](x) \doteq \frac{u(x) - l(x)}{\sum_y (u(y) - l(y))} = \frac{\Delta(x)}{\sum_y \Delta(y)}$$

- $\Delta(x)$  measures the size of the probability interval on  $x$
- $R(x)$  measures how much the width of the interval for  $x$  “weights” on the total width of the set of probability intervals
- relative uncertainty of singletons**

## Example

- same set of intervals as before

$$0.2 \leq p(x) \leq 0.8, \quad 0.4 \leq p(y) \leq 1, \quad 0 \leq p(z) \leq 0.4$$

- common fraction of the intervals

$$\beta = \frac{1 - 0.2 - 0.4 - 0}{0.6 + 0.6 + 0.4} = \frac{0.4}{1.6} = \frac{1}{4}$$

- values of the intersection probability

$$p[(l, u)](x) = 0.2 + \frac{1}{4}0.6 = 0.35, \quad p[(l, u)](y) = 0.4 + \frac{1}{4}0.6 = 0.55, \\ p[(l, u)](z) = 0 + \frac{1}{4}0.4 = 0.1$$

# Intersection probability for BFs

- each belief measure also determines a set of probability intervals
- $\rightarrow$  the intersection probability can be defined for BFs too
- intervals associated with a BF  $b(A) = \sum_{B \subseteq A} m_b(B)$

$$(b, pl_b) \doteq \{p \in \mathcal{P} : b(x) \leq p(x) \leq pl_b(x), \forall x \in \Theta\}$$

- intersection prob for BFs: replace

$$l(x) \rightarrow b(x), \quad u(x) \rightarrow pl_b(x)$$

in the expression for generic prob intervals

# Credal interpretation of BFs

## and pignistic transform

- each belief function determines an entire set of probabilities *consistent* with it

$$\mathcal{P}[b] \doteq \left\{ p \in \mathcal{P} : b(A) \leq p(A) \leq pl_b(A) \quad \forall A \subseteq \Theta \right\}$$

- different* from the set of probabilities determined by the probability interval  $(b, pl_b)$
- it is a convex set (credal set), with barycenter

$$BetP[b](x) = \sum_{A \ni \{x\}} \frac{m_b(A)}{|A|}$$

- similar credal interpretation for the intersection probability?**

# Credal interpretation of interval probs

## Upper and lower simplices

- consider the set of probabilities which meet the **lower** constraint *on singletons*  $T^1[b]$ :

$$T^1[b] \doteq \left\{ p \in \mathcal{P} : p(x) \geq b(x) \quad \forall x \in \Theta \right\}$$

- and the set of probs meeting the **upper** constraint *on singletons*

$$T^{n-1}[b] \doteq \left\{ p \in \mathcal{P} : p(x) \leq pl_b(x) \quad \forall x \in \Theta \right\}$$

- the pair  $(T^1[b], T^{n-1}[b])$  is the **geometric counterpart of an interval probability** in the probability simplex

# Upper and lower simplices

- the form two higher-dimensional triangles or **simplices**
- $T^1[b] = Cl(t_x^1[b], x \in \Theta)$  with vertices

$$t_x^1[b] = \sum_{y \neq x} m_b(y) b_y + (1 - \sum_{y \neq x} m_b(y)) b_x$$

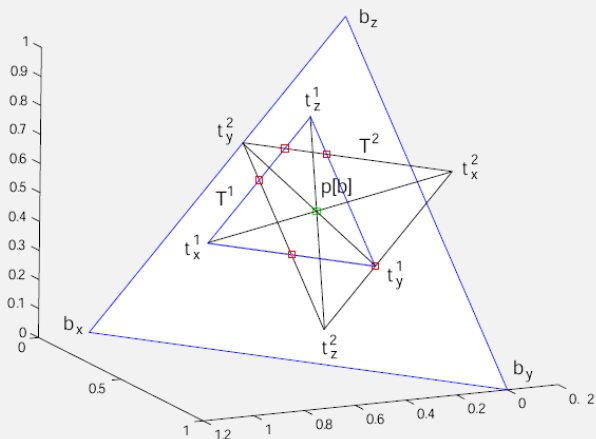
- $T^{n-1}[b] = Cl(t_x^{n-1}[b], x \in \Theta)$  with vertices

$$t_x^{n-1}[b] = \sum_{y \neq x} pl_b(y) b_y + (1 - \sum_{y \neq x} pl_b(y)) b_x$$

# Ternary example

- example: the case of a BF on  $\Theta = \{x, y, z\}$ :

$$\begin{array}{lll}
 m_b(x) = 0.2, & m_b(y) = 0.1, & m_b(z) = 0.3, \\
 m_b(\{x, y\}) = 0.1, & m_b(\{y, z\}) = 0.2, & m_b(\Theta) = 0.1
 \end{array}$$



# Focus of a pair of simplices

- $p[(l, u)]$  unique **intersection of the lines joining corresponding vertices** of upper and lower simplices
- can be formalized by the notion of “focus”
- **focus** of  $S, T$ : unique point with the **same affine coordinates** in both simplices

$$f(S, T) = \sum_{i=1}^n \alpha_i s_i = \sum_{j=1}^n \alpha_j t_j, \quad \sum_{i=1}^n \alpha_i = 1$$

- does not always fall in their intersection  $S \cap T$
- when it does, it coincides with intersection of lines passing through vertices

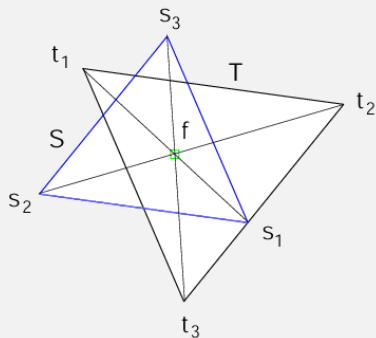


# Intersection probability as focus

of upper and lower simplices

## Theorem

*The intersection probability is the focus of the pair of upper and lower simplices:  $p[b] = f(T^{n-1}[b], T^1[b])$ .*



- interpretation: only probability meeting both constraints **in exactly the same way**
- **similar interpretation holds for relative belief and plausibility** [Cuzzolin, ISIPTA'09]

# Affine family

## of probability transformations

- as a transformation of BFs, intersection probability belongs to the **affine family**
- group of transformations which *commute with affine combination* (at least under certain conditions)

### Theorem

*Intersection probability  $p[b]$  and affine combination commute:*

$$p[\alpha_1 b_1 + \alpha_2 b_2] = \alpha_1 p[b_1] + \alpha_2 p[b_2]$$

*for  $\alpha_1 + \alpha_2 = 1$ , if the relative uncertainty of the singletons is the same for both intervals:  $R[b_1] = R[b_2]$*

# Comparison

with other members of the affine family

- other members of the affine family: *pignistic transform*  $BetP[b]$  [Smets] and *orthogonal projection*  $\pi[b]$  of  $b$  onto  $\mathcal{P}$  [Cuzzolin, SMC-B'07]
- conditions under which they reduce to the same transformation?

## Theorem

If a belief function  $b$  is such that its mass is **equally distributed among focal elements of the same size**

$$m_b(A) = \text{const} \quad \forall A : |A| = k, \quad \forall k = 2, \dots, n$$

then its pignistic transform, intersection probability, and orthogonal projection coincide:  $BetP[b] = p[b] = \pi[b]$ .

# Conclusions

- probability transformation studied for BFs
- intersection probability: rational transformation *for sets of prob intervals*
- treats *all elements of the frame in the same way*
- this reflects in an elegant geometry in terms of focus of upper and lower simplices
- meets *upper and lower constraint in the same way*
- focus interpretation **extends to many other transformations**
- step towards a comprehensive classification of all probability transformations