Complexes of outer consonant approximations

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Outline

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Belief functions

- basic belief assignment $m : 2^\Theta \to [0,1]$ such that

$$m(\emptyset) = 0, \sum_{A \subseteq \Theta} m(A) = 1, m(A) \geq 0 \ \forall A \subseteq \Theta$$

- **belief function** $b : 2^\Theta \to [0,1]$,

$$b(A) = \sum_{B \subseteq A} m(B)$$

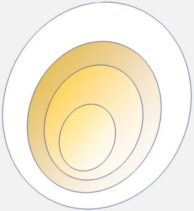
- a BF is described by its list of **focal elements**, subsets of Θ with non-zero mass

$$\{A \subseteq \Theta : m(A) \neq 0\}$$

Consonant belief functions

- some special classes of BFs:
- probability measures or Bayesian b.f.s: $m_b(A) = 0, |A| > 1$
- necessity measures of **consonant belief functions**: BFs whose

focal elements are *nested*



Ordering belief functions

- BFs admit the following order relation

$$b \leq b' \quad \equiv \quad b(A) \leq b'(A) \qquad \forall A \subseteq \Theta$$

- set of probabilities greater than b: **consistent probabilities** $\mathcal{P}[b]$, i.e. those meeting the lower bound given by b
  - very well studied, foundation for Transferable Belief Model [Smets]
  - $\mathcal{P}[b]$ is a polytope [Jaffray'89]
  - center of mass is the pignistic transform $BetP[b]$ [Smets]

- **what about consonant BFs?** consonant BFs smaller than b?

Outer consonant approximations of BFs

- **outer consonant approximations** [Dubois'90] of a belief function b as those COBFs such that

$$co(A) \leq b(A) \quad \forall A \subseteq \Theta$$

- interesting to look for *minimal* approximation, with respect to the weak inclusion relation
- in particular a family of approximations generated by element permutations has been proposed

Outer consonant approximations of BFs

A family of minimal approximations

- consider all permutations ρ of the elements $\{x_1, ..., x_n\}$ of the frame of discernment Θ: $\{x_{\rho(1)}, ..., x_{\rho(n)}\}$
- any permutation ρ generates a family of nested sets ...

$$\Big\{ S_1^\rho = \{x_{\rho(1)}\}, S_2^\rho = \{x_{\rho(1)}, x_{\rho(2)}\}, ..., S_n^\rho = \{x_{\rho(1)}, ..., x_{\rho(n)}\} \Big\}$$

- ... so it induces a COBF co^ρ with mass assignment

$$m_{co^\rho}(S_j^\rho) = \sum_{i:\min\{l:E_i \subseteq S_l^\rho\}=j} m_b(E_i). \quad (1)$$

- S_j^ρ is assigned the mass of the focal elements of b included in S_j^ρ but not in S_{j-1}^ρ

Goals of our investigation

Goal

Study the set of outer consonant approximations of belief functions.

Goal

Understand the geometry of this set. Is it convex? What are its vertices?

Goal

Understand where are located the minimal approximation. Do they coincide with some of the vertices?

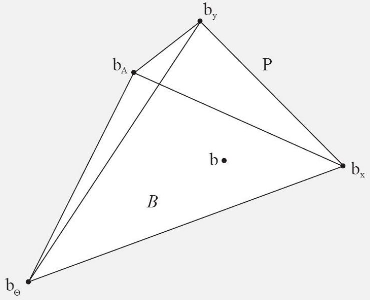
Geometric approach to uncertainty

- belief functions $b : 2^\Theta \rightarrow [0,1]$ are completely specified by their $N-2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$, $N \doteq 2^{|\Theta|}$
- they can then be represented as points of $\mathbb{R}^{N-2}$

- belief functions form a simplex

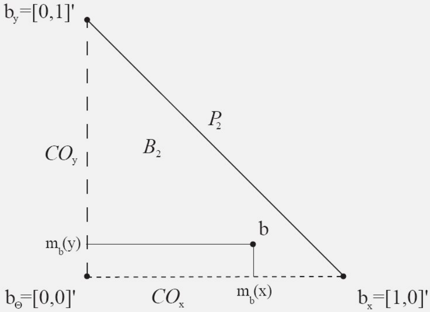
$$\mathcal{B} = Cl(b_A, \emptyset \subsetneq A \subsetneq \Theta);$$

- vertices b_A: *categorical* BFs : $b_A(A) = 1$



Example: the binary case

- each b.f. $b : 2^{\Theta_2} \rightarrow [0,1]$ corresponds to a vector $[b(x) = m_b(x), b(y) = m_b(y)]'$;



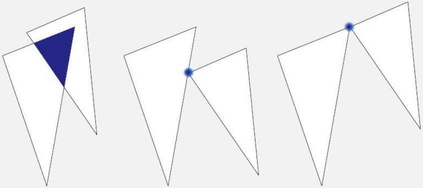
$$\mathcal{CO}_2 = \mathcal{CO}_x \cup \mathcal{CO}_y = Cl(b_\Theta, b_x) \cup Cl(b_\Theta, b_y)$$

Simplicial complexes

Definition

A *simplicial complex* is a collection Σ of simplices which satisfies the following properties:

1. if a simplex belongs to Σ, then all its faces belong to Σ
2. the intersection of any two simplices is a face of both



The consonant complex

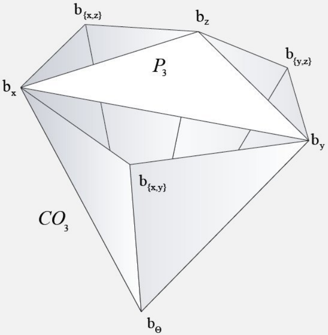
Proposition

The region $\mathcal{CO}$ of consonant belief functions in the belief space is a simplicial complex.

- consonant subspace

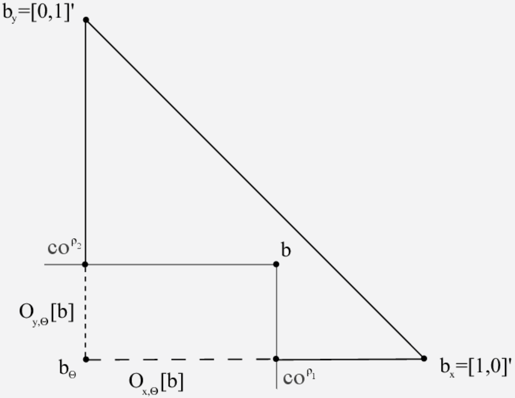
$$\mathcal{CO} = \bigcup_{A_1 \subset ... \subset A_n} Cl(b_{A_1}, ..., b_{A_n});$$

- each maximal simplex is associated with a maximal chain of sets $A_1 \subset ... \subset A_n$;



Consonant approx in the binary case

- illustration of outer consonant approxs for $\Theta = \{x, y\}$



- we can conjecture a few facts on the general case (continue)

Conjectures induced by the binary case

- the set of outer cons. approx $O[b]$ is a collection of sets $O_{\mathcal{C}}[b]$, one for each maximal chain
- for each maximal chain $\mathcal{C}$ of focal elements:

  1. $O_{\mathcal{C}}[b]$ is convex
  2. $O_{\mathcal{C}}[b]$ is a *polytope*, i.e. the convex closure of a number of vertices (a segment in the binary case, $O_{x,\Theta}[b]$ or $O_{y,\Theta}[b]$)
  3. the maximal outer approximation of b is one of the vertices of this polytope $O_{\mathcal{C}}[b]$
  4. the latter is the vertex co^{ρ} associated with the permutation ρ of singletons which generates the chain $\mathcal{C}$

- all these properties indeed **hold in the general case**

Vertices of the simplex of outer approxs

... and assignment functions

- the vertices of $O_{\mathcal{C}}[b]$ are all the COBFs $o^{\vec{B}}[b]$ with b.p.a.

$$m_{o^{\vec{B}}[b]}(B_i) = \sum_{A \subseteq \Theta : \vec{B}(A) = B_i} m_b(A) \quad (2)$$

- each of them is associated with an "assignment function"

$$\begin{array}{rcccl} \vec{B} & : & 2^{\Theta} & \rightarrow & \mathcal{C} \\ & & A & \mapsto & \vec{B}(A) \supseteq A \end{array} \quad (3)$$

- they map each event A to one of the events of the chain $\mathcal{C} = \{B_1 \subset ... \subset B_n\}$ which contains A

Ternary case : assignment functions

- all assignment functions for $\Theta = \{x, y, z\}$
- notation:

$$\vec{B} = [\vec{B}(\{x\}), \vec{B}(\{y\}), \vec{B}(\{z\}), \vec{B}(\{x,y\}), \vec{B}(\{x,z\}), \vec{B}(\{y,z\}), \vec{B}(\{x,y,z\})]$$

- example of a few assignment functions:

$$\begin{array}{lllllllll}
\vec{B}_1 & = [& \{x\}, & \{x,y\}, & \Theta, & \{x,y\}, & \Theta, & \Theta, & \Theta \quad]; \\
\vec{B}_2 & = [& \{x\}, & \{x,y\}, & \Theta, & \Theta, & \Theta, & \Theta, & \Theta \quad]; \\
\vec{B}_3 & = [& \{x\}, & \Theta, & \Theta, & \{x,y\}, & \Theta, & \Theta, & \Theta \quad]; \\
\vec{B}_4 & = [& \{x\}, & \Theta, & \Theta, & \Theta, & \Theta, & \Theta, & \Theta \quad]; \\
\vec{B}_5 & = [& \{x,y\}, & \{x,y\}, & \Theta, & \{x,y\}, & \Theta, & \Theta, & \Theta \quad]; \\
\vec{B}_6 & = [& \{x,y\}, & \{x,y\}, & \Theta, & \Theta, & \Theta, & \Theta, & \Theta \quad]; \\
\vec{B}_7 & = [& \ldots & & & & & &
\end{array}$$

Main results

Theorem

For each simplicial component $\mathcal{CO}_\mathcal{C}$ of the consonant space associated with any maximal chain of focal elements $\mathcal{C} = \{B_1, ..., B_n\}$ the set of outer consonant approximation of any b.f. b is the convex closure

$$O_\mathcal{C}[b] = Cl(o^{\vec{B}}[b], \forall \vec{B})$$

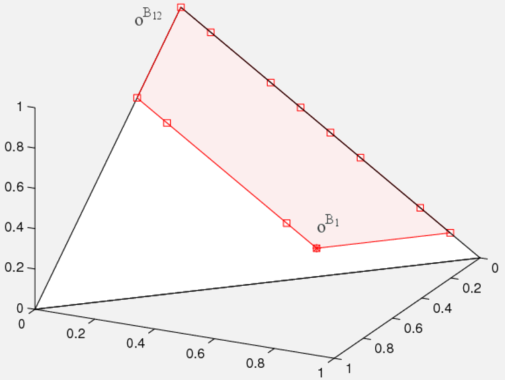
of the COBFs (2) indexed by all admissible assignment functions (3).

Theorem

The maximal outer consonant approximation with maximal chain $\mathcal{C}$ of a belief function b is the vertex (1) of $O_{\mathcal{C}_\rho}[b]$ associated with the permutation ρ of the singletons which generates $\mathcal{C} = \mathcal{C}_\rho$.

Ternary case : polytope of approximations

- for the chain $\mathcal{C} = \{\{x\}, \{x, y\}, \{x, y, z\}\}$



- not all the assignment functions produce a true vertex
- minimal approximation $o^{B_{12}}$, generated by the permutation $\rho = (x, y, z)$

Conclusions

- outer consonant approxs, dual of inner Bayesian approxs (consistent BFs)
- the latter form a simplex, the former a set of polytopes, one for each maximal chain of focal elements
- possible vertices generated by all assignment functions
- minimal approx vertex determined by element permutation
- natural extension to consistent BFs