Supplement for the paper

Monadic Maps and Folds for Multirelations in an Allegory

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This document contains supplementary justifications, proofs and references for results in the main paper.

Several relational and multirelational algebraic laws (see below) are required for subsequent proofs, and these laws are proved at the end of this supplement. In each case, r and s denote arbitrary arrows in the allegory and f, g, h and k denote function arrows. Let Λ^{-1} be defined on functions by $\Lambda^{-1} f = f \,;\ni$.

$$\Lambda r \,;\ni\; = r \tag{S.1}$$

$$\Lambda(f \,;\ni) = f \tag{S.2}$$

$$\Lambda(f \,; r) = f \,; \Lambda r \tag{S.3}$$

$$\mathsf{E} r \,;\ni\; = \;\ni ; r \tag{S.4}$$

$$\Lambda(r \,; s) = \Lambda r \,; \mathsf{E} s \tag{S.5}$$

$$\Lambda(\Lambda^{-1} f) = f \tag{S.6}$$

$$\Lambda^{-1}(\Lambda r) = r \tag{S.7}$$

$$r\backslash(f \,; s) = (f^\circ \,; r)\backslash s \tag{S.8}$$

$$f \,; (r\backslash s) = (r \,; f^\circ)\backslash s \tag{S.9}$$

$$\mathsf{E} f \,; (\in \backslash r) = \;\in \backslash(f \,; r) \tag{S.10}$$

$$(t\backslash s) \,; (s\backslash r) \subseteq t\backslash r \tag{S.11}$$

$$id\backslash r = r \tag{S.12}$$

$$s \,; (s\backslash r) \subseteq r \tag{S.13}$$

$$(r\backslash(s \,; t)) \,; (t\backslash u) \subseteq r\backslash(s \,; u) \tag{S.14}$$

$$(r\backslash \in) \,; (\in \backslash s) = r\backslash s \tag{S.15}$$

$$(r\backslash s) \times (t\backslash u) \subseteq (r \times t)\backslash(s \times u) \tag{S.16}$$

$$(\mathsf{E} f \times \mathsf{E} g) \,; ((\in \times \in)\backslash r) = (\in \times \in)\backslash((f \times g) \,; r) \tag{S.17}$$

$$(r\backslash s) \,; t \subseteq r\backslash(s \,; t) \tag{S.18}$$

$$(f \star g) \,;\ni\; = f \,;\ni ; (\in \backslash(g \,;\ni)) \tag{S.19}$$

$$\begin{aligned}(f \star g \;\times\; h \star k) \,; (\ni \times \ni) \subseteq\; & (f \times h) \,;\; (\ni \times \ni) \,; \\ & ((\in \times \in)\backslash((g \times k) \,; (\ni \times \ni)))\end{aligned} \tag{S.20}$$

$$(f \,;_{\mathsf{M}} g) \,; e = (f \,; e) \star (g \,; e) \tag{S.21}$$

Also note that if there exists a function p such that $r \,; p^\circ = id$, then

$$r \,; (r\backslash t) = t \tag{S.22}$$

Proofs of Claims in the Main Paper

Claim. At the end of Section 3, it is claimed that $(r\,;s) \fatsemi t \;=\; r\,;(s \fatsemi t)$. To be clear about the types, this is for any arrows $r : W \to X$, $s : X \rightrightarrows Y$ and $t : Y \rightrightarrows Z$ in a power allegory:

Proof.

$$
\begin{array}{cl}
 & (r\,;s) \fatsemi t \\
= & \{\text{Definition 3.2 (of } \fatsemi)\} \\
 & (r\,;s)\,;(\in \backslash t) \\
= & \{\text{Associativity of } ;\} \\
 & r\,;(s\,;(\in \backslash t)) \\
= & \{\text{Definition of } \fatsemi\} \\
 & r\,;(s \fatsemi t)
\end{array}
$$

□

Claim. In Definition 3.4 it is claimed that for allegory A, $\mathbf{Mul}(\mathsf{A})$ is a category. This is shown in [18]. □

Claim. Lemma 4.4 claims that $\mathbf{MFun}(\mathsf{A})$ is an order-enriched category, and Lemma 4.5 claims that $\mathbf{MFun}(\mathsf{A}) \cong \mathbf{Mul}(\mathsf{A})$. We will prove these together:

Proof. $\mathbf{MFun}(\mathsf{A})$ is an order-enriched category if the following properties hold:

- composition is associative
- composition preserves the up-closure property (Definition 4.1)
- ι is the identity of composition
- composition is monotonic with respect to $\leq$

These properties will not be proved explicitly here, since they have already been proved for $\mathbf{Mul}(\mathsf{A})$ in [18], and thus it is sufficient to show that there is an isomorphism between up-closed multirelations and up-closed multifunctions with the following properties:

1. identity-preserving
2. distributes through composition
3. order-preserving on homsets

The function that maps multirelations to multifunctions is the power transpose Λ, which has inverse Λ^{-1} as shown by (S.6) and (S.7). To see that Λ maps multirelations to multifunctions, note that by the definition of $\uparrow$ (see Definition 4.1) and equation (S.5), if $r : A \rightrightarrows B$ then Λr satisfies the up-closure property (Definition 4.1) and is therefore a up-closed multifunction. Conversely, suppose

$p : A \to \mathsf{P}^2 B$ satisfies the up-closure property (Definition 4.1), then the multirelation $\varLambda^{-1}p$ is up-closed, since

$$
\begin{array}{ll}
& p\,;\ni;\sqsubseteq \\
= & \{(\text{S.4})\} \\
& p\,;(\mathsf{E}\sqsubseteq)\,;\ni \\
= & \{\text{Definition 4.1, including the definition of } \uparrow\} \\
& p\,;\ni
\end{array}
$$

It remains to show that this isomorphism satisfies the three properties listed above. The first property is straightforward from the definitions of ι (Definition 4.2) and $\varLambda^{-1}$, and the last property is straightforward from the definitions of $\varLambda^{-1}$ and the ordering $\leq$ (see Lemma 4.4), and these proofs are omitted. It remains to show that the isomorphism distributes through composition. It is sufficient to establish this property for $\varLambda$ since it can then be deduced for $\varLambda^{-1}$ by the mutual inverse property. So let $r, s \in \mathbf{Mul}(\mathsf{A})$:

$$
\begin{array}{ll}
& \varLambda(r \fatsemi s) \\
= & \{\text{Let } p = \varLambda r \text{ and } q = \varLambda s\} \\
& \varLambda(\varLambda^{-1}p \fatsemi \varLambda^{-1}q) \\
= & \{\text{Definition of } \varLambda^{-1}\} \\
& \varLambda((p\,;\ni) \fatsemi (q\,;\ni)) \\
= & \{\text{Definition of } \fatsemi\} \\
& \varLambda((p\,;\ni)\,;(\in \backslash (q\,;\ni))) \\
= & \{(\text{S.10})\} \\
& \varLambda((p\,;\ni)\,;\mathsf{E}q\,;(\in \backslash \ni)) \\
= & \{(\text{S.5})\} \\
& \varLambda(p\,;\ni)\,;\mathsf{E}(\mathsf{E}q\,;(\in \backslash \ni)) \\
= & \{\text{Claim below, and } \mathsf{E} \text{ is a functor}\} \\
& p\,;\mathsf{E}^2 q\,;\mathsf{E}(\in \backslash \ni)) \\
= & \{(\text{S.23})\} \\
& p\,;\mathsf{E}^2 q\,;\mathsf{E}\cap;\cup \\
= & \{\text{Definition 4.3 (of } \star)\} \\
& p \star q \\
= & \{\text{Definition of } p \text{ and } q\} \\
& \varLambda r \star \varLambda s
\end{array}
$$

The claim above was

$$\mathsf{E}\cap;\cup = \mathsf{E}(\in \backslash \ni) \tag{S.23}$$

and is proved:

$$
\begin{array}{cl}
& \mathsf{E}\cap\,;\cup \\
= & \{\text{Definitions of } \cap \text{ and } \cup\} \\
& \mathsf{E}(\Lambda(\in \setminus \ni))\,;\Lambda(\ni\,;\ni) \\
= & \{\text{(S.3) and (S.4)}\} \\
& \Lambda(\ni\,;\Lambda((\in \setminus \ni)\,;\ni) \\
= & \{\text{(S.1)}\} \\
& \Lambda(\ni\,;(\in \setminus \ni)) \\
= & \{\text{Definition 2.7 (of } \mathsf{E})\} \\
& \mathsf{E}(\in \setminus \ni)
\end{array}
$$

So we have shown that the isomorphism defined by Λ and Λ^{-1} satisfies the required three properties. We can therefore deduce that the composition operator on multifunctions is therefore associative with identity ι, monotonic and it preserves the up-closure property (see Definition 4.1) because the corresponding properties are known to hold of multirelational composition. Therefore there is an order-isomorphism of categories $\mathbf{Mul}(\mathsf{A}) \cong \mathbf{MFun}(\mathsf{A})$. □

Claim. Equation (4.3) claims that for any $f : X \to Y$ and $m : Y \to \mathsf{P}^2 Z$, we have $\widehat{f} \star m \;=\; f\,;m$.

Proof.

$$
\begin{array}{cl}
& \widehat{f} \star m \\
= & \{\text{Definition 4.6 (of } \widehat{\ }\text{)}\} \\
& (f\,;\iota) \star m \\
= & \{\text{Equation (4.1) }\} \\
& f\,;(\iota \star m) \\
= & \{\text{Identity of } \star\} \\
& f\,;m
\end{array}
$$

□

Claim. Equation (4.4) claims that for any $f : X \to Y$ and $g : Y \to Z$, we have $\widehat{f} \star \widehat{g} \;=\; \widehat{f\,;g}$.

Proof.

$$
\begin{array}{cl}
& \widehat{f} \star \widehat{g} \\
= & \{\text{Equation (4.3), see above}\}
\end{array}
$$

$$
\begin{aligned}
& f\,;\widehat{g} \\
= \quad & \{\text{Definition 4.6 (of } \widehat{\ }\text{)}\} \\
& f\,;g\,;\iota \\
= \quad & \{\text{Definition of } \widehat{\ }\} \\
& \widehat{f\,;g}
\end{aligned}
$$

□

Claim. Equation (4.4) claims that for any $m : X \to \mathsf{P}^2 Y$ and $f : Y \to Z$, we have $m \star \widehat{f} \;=\; m\,;\mathsf{E}^2 f\,;\uparrow$.

Proof. First, we observe that for any multifunction $g : X \to \mathsf{P}^2 Y$ (not-necessarily up-closed),

$$g \star \iota = g\,;\uparrow \tag{S.24}$$

since

$$
\begin{aligned}
& g \star \iota \\
= \quad & \{\text{Definition 4.3 (of } \star\text{); (S.23)}\} \\
& g\,;\mathsf{E}^2\iota\,;\mathsf{E}(\in \setminus \ni) \\
= \quad & \{\mathsf{E} \text{ is a functor}\} \\
& g\,;\mathsf{E}(\mathsf{E}\iota\,;(\in \setminus \ni)) \\
= \quad & \{\text{(S.10)}\} \\
& g\,;\mathsf{E}(\in \setminus(\iota\,;\ni)) \\
= \quad & \{\text{Definition 4.2 (of } \iota\text{); (S.1)}\} \\
& g\,;\mathsf{E}(\in \setminus \in) \\
= \quad & \{\text{Definition of } \uparrow \text{ (see Definition 4.1)}\} \\
& g\,;\uparrow
\end{aligned}
$$

Then we can calculate:

$$
\begin{aligned}
& m \star \widehat{f} \\
= \quad & \{\text{Definition 4.6 (of } \widehat{\ }\text{) }\} \\
& m \star (f\,;\iota) \\
= \quad & \{\text{Equation (4.2) }\} \\
& (m\,;\mathsf{E}^2 f) \star \iota \\
= \quad & \{\text{(S.24 }\} \\
& (m\,;\mathsf{E}^2 f)\,;\uparrow
\end{aligned}
$$

□

Claim. Equation (4.6) claims that for any $f : X \to Y$, we have $\widehat{f} \,;\ni \;=\; f \,;\in$.

Proof.

$$\begin{aligned}
& \widehat{f} \,;\ni \\
= \quad & \{\text{Definitions 4.6 (of } \widehat{\ }) \text{ and 4.2 (of } \iota)\} \\
& f \,;\Lambda \in \,;\ni \\
= \quad & \{(\text{S.1})\} \\
& f \,;\in
\end{aligned}$$

□

Claim. Lemma 5.8 was proved in the appendix of the main paper. □

Claim. Lemma 5.9 claimed that $\langle \mathsf{N}, \eta, \mu \rangle$ is a monad.

Proof. It is necessary to show that both η and μ are natural transformations and that the monad laws in Definition 5.1 are satisfied. For the naturality condition on η, by the equivalence (A.2) it is sufficient to show that for all $p : X \to Y$ in C, it is the case that $\eta_X \,; \mathsf{N}p \,; e_Y = p \,; \eta_Y \,; e_Y$.

$$\begin{aligned}
& \eta_X \,; \mathsf{N}\, p \,; e_Y \\
= \quad & \{\text{Definition 5.7}\} \\
& \eta_X \,; (e_X \star \widehat{p}) \\
= \quad & \{\text{Equation (4.1)}\} \\
& (\eta_X \,; e_X) \star \widehat{p} \\
= \quad & \{\text{Definition of } \eta\} \\
& \iota_X \star \widehat{p} \\
= \quad & \{\text{Identity of } \star\} \\
& \widehat{p} \\
= \quad & \{\text{Definition 4.6 (of } \widehat{\ }) \text{ and the definition of } \eta\} \\
& p \,; \eta_Y \,; e_Y
\end{aligned}$$

For the naturality condition on μ, by the equivalence (A.2) it is sufficient to show that for all $p : X \to Y$ in C, $\mathsf{N}^2 p \,; \mu_Y \,; e_Y = \mu_X \,; \mathsf{N}p \,; e_Y$.

$$\begin{aligned}
& \mathsf{N}^2 p \,; \mu_Y \,; e_Y \\
= \quad & \{\text{Definition 5.7 and the definition of } \mu\} \\
& (e_{\mathsf{N}X} \star \widehat{\mathsf{N}p})' \,; (e_{\mathsf{N}Y} \star e_Y)' \,; e_Y \\
= \quad & \{\text{Equation (A.1)}\} \\
& e_{\mathsf{N}X} \star \widehat{\mathsf{N}p} \star e_Y
\end{aligned}$$

$$
\begin{aligned}
&= \quad \{\text{Equation (4.3)}\} \\
&\quad e_{\mathsf{N}X} \star (\mathsf{N}p \,; e_Y) \\
&= \quad \{\text{Definition 5.7 and associativity of } \star\} \\
&\quad e_{\mathsf{N}X} \star e_X \star \widehat{p} \\
&= \quad \{\text{Definition 5.7, definition of } \mu \text{ and equation (A.1)}\} \\
&\quad \mu_X \,; \mathsf{N}p \,; e_Y
\end{aligned}
$$

Now we prove the monad laws: by the equivalence (A.2),

$$\mathsf{N}\ \eta_X \,; \mu_X = id \quad \equiv \quad \mathsf{N}\ \eta_X \,; \mu_X \,; e_X = e_X$$

and we can calculate

$$
\begin{aligned}
&\quad \mathsf{N}\,\eta_X \,; \mu_X \,; e_X \\
&= \quad \{\text{Definition of } \mathsf{N} \text{ (Definitions 5.5 and 5.7) and } \mu\} \\
&\quad (e_X \star \widehat{\eta_X})' \,; (e_{\mathsf{N}X} \star e_X)' \,; e_X \\
&= \quad \{\text{Equation (A.1)}\} \\
&\quad e_X \star \widehat{\eta_X} \star e_X \\
&= \quad \{\text{Equation (4.3)}\} \\
&\quad e_X \star (\eta_X \,; e_X) \\
&= \quad \{\text{Definition of } \eta\} \\
&\quad e_X \star \iota_X \\
&= \quad \{\text{Identity of } \star\} \\
&\quad e_X
\end{aligned}
$$

For the second monad law, by the equivalence (A.2) we have that

$$\eta\mathsf{N}_X \,; \mu_X = id \quad \equiv \quad \eta\mathsf{N}_X \,; \mu_X \,; e_X = e_X$$

and we can calculate

$$
\begin{aligned}
&\quad \eta\mathsf{N}_X \,; \mu_X \,; e_X \\
&= \quad \{\text{Definition of } \mu\} \\
&\quad \eta\mathsf{N}_X \,; (e_{\mathsf{N}X} \star e_X) \\
&= \quad \{\text{Equation (4.1)}\} \\
&\quad (\eta\mathsf{N}_X \,; e_{\mathsf{N}X}) \star e_X \\
&= \quad \{\text{Definition of } \eta\} \\
&\quad \iota \star e_X \\
&= \quad \{\text{Identity of } \star\} \\
&\quad e_X
\end{aligned}
$$

For the last monad law, by the equivalence (A.2):

$$\mu\mathsf{N}_X \,;\, \mu_X = \mathsf{N}\mu_X \,;\, \mu_X \quad \equiv \quad \mu\mathsf{N}_X \,;\, \mu_X \,;\, e_X = \mathsf{N}\mu_X \,;\, \mu_X \,;\, e_X$$

and we can calculate

$$\begin{aligned}
& \mathsf{N}\mu_X \,;\, \mu_X \,;\, e_X \\
= \quad & \{\text{Definition 5.7 and the definition of } \mu\} \\
& (e_{\mathsf{N}^2X} \star \widehat{\mu}_X)' \,;\, (e_{\mathsf{N}X} \star e_X)' \,;\, e_X \\
= \quad & \{\text{Equation (A.1) and associativity of } \star\} \\
& e_{\mathsf{N}^2X} \star \widehat{\mu}_X \star e_X \\
= \quad & \{\text{Equation (4.3)}\} \\
& e_{\mathsf{N}^2X} \star (\mu_X \,;\, e_X) \\
= \quad & \{\text{Definition of } \mu\} \\
& e_{\mathsf{N}^2X} \star e_{\mathsf{N}X} \star e_X) \\
= \quad & \{\text{Equation (A.1)}\} \\
& (e_{\mathsf{N}^2X} \star e_{\mathsf{N}X})' \,;\, (e_{\mathsf{N}X} \star e_X)' \,;\, e_X \\
= \quad & \{\text{Definition of } \mu\} \\
& \mu_{\mathsf{N}X} \,;\, \mu_X \,;\, e_X
\end{aligned}$$

□

Claim. Theorem 5.10 stated that there is an order-isomorphism of categories $\mathbf{Fun}(\mathsf{A})^{\mathsf{N}} \cong \mathbf{MFun}(\mathsf{A})$.

Proof. The definitions of identity and ordering in $\mathbf{Fun}(\mathsf{A})^{\mathsf{N}}$ correspond to those in $\mathbf{MFun}(\mathsf{A})$ by the isomorphism defined by the universal property given in Definition 5.6. So it just remains to check that the isomorphism distributes through composition, which is shown as follows:

$$\begin{aligned}
& p \star q \\
= \quad & \{\text{Universal property from Definition 5.6}\} \\
& (p' \,;\, e_Y) \star (q' \,;\, e_Z) \\
= \quad & \{\text{Equation (4.3)}\} \\
& (p' \,;\, e_Y) \star \widehat{q'} \star e_Z \\
= \quad & \{\text{Equation (4.1)}\} \\
& p' \,;\, (e_Y \star \widehat{q'}) \star e_Z \\
= \quad & \{\text{Definition 5.7 }\} \\
& (p' \,;\, \mathsf{N}(q') \,;\, e_{\mathsf{N}Z}) \star e_Z \\
= \quad & \{\text{Equation (4.1)}\} \\
& p' \,;\, \mathsf{N}(q') \,;\, (e_{\mathsf{N}Z} \star e_Z)
\end{aligned}$$

$$= \quad \{\text{Definition of } \mu\}$$
$$p' \,;\, \mathsf{N}(q') \,;\, \mu_Z \,;\, e_Z$$
$$= \quad \{\text{Definition of } ;_{\mathsf{N}}\}$$
$$(p' \,;_{\mathsf{N}}\, q') \,;\, e_Z$$

□

Claim. Lemma 6.9 stated that a lax lifting F^{M} preserves identities and that $\mathsf{F}^{\mathsf{M}}(p \,;_{\mathsf{M}}\, q) \preceq \mathsf{F}^{\mathsf{M}}p \,;_{\mathsf{M}}\, \mathsf{F}^{\mathsf{M}}q$ for all $p : X \to Y$ and $q : Y \to Z$ in category C^{M}.

Proof. First we show that F^{M} preserves identities:

$$\mathsf{F}^{\mathsf{M}}\eta_X$$
$$= \quad \{\text{Definition of } \mathsf{F}^{\mathsf{M}} \text{ (see Definition 6.8)}\}$$
$$\mathsf{F}\eta_X \,;\, \delta^{\mathsf{F}}_X$$
$$= \quad \{\text{Equation (6.5)}\}$$
$$\eta_{\mathsf{F}X}$$

For the other condition, note that the arrows p, q in the category C^{M} are also arrows $p : X \to \mathsf{M}Y$ and $q : Y \to \mathsf{M}Z$ in category C. Then we calculate:

$$\mathsf{F}^{\mathsf{M}}(p \,;_{\mathsf{M}}\, q)$$
$$= \quad \{\text{Definition of } \mathsf{F}^{\mathsf{M}}\}$$
$$\mathsf{F}(p \,;_{\mathsf{M}}\, q) \,;\, \delta^{\mathsf{F}}_Z$$
$$= \quad \{\text{Definition of } ;_{\mathsf{M}} \text{ (see Definition 5.2)}\}$$
$$\mathsf{F}(p \,;\, \mathsf{M}q \,;\, \mu_Z) \,;\, \delta^{\mathsf{F}}_Z$$
$$= \quad \{\mathsf{F} \text{ is a functor}\}$$
$$\mathsf{F}p \,;\, \mathsf{FM}q \,;\, \mathsf{F}\mu_Z \,;\, \delta^{\mathsf{F}}_Z$$
$$\preceq \quad \{\text{Equation (6.6)}\}$$
$$\mathsf{F}p \,;\, \mathsf{FM}q \,;\, (\delta^{\mathsf{F}}_{\mathsf{M}Z} \,;_{\mathsf{M}}\, \delta^{\mathsf{F}}_Z)$$
$$= \quad \{\text{Definition of } ;_{\mathsf{M}}\}$$
$$\mathsf{F}p \,;\, \mathsf{FM}q \,;\, \delta^{\mathsf{F}}_{\mathsf{M}Z} \,;\, \mathsf{M}\delta^{\mathsf{F}}_Z \,;\, \mu_Z$$
$$\preceq \quad \{\text{Equation (6.4)}\}$$
$$\mathsf{F}p \,;\, \delta^{\mathsf{F}}_Y \,;\, \mathsf{MF}q \,;\, \mathsf{M}\delta^{\mathsf{F}}_Z \,;\, \mu_Z$$
$$= \quad \{\text{Definition of } \mathsf{F}^{\mathsf{M}} \text{ and } \mathsf{M} \text{ is a functor}\}$$
$$\mathsf{F}^{\mathsf{M}}p \,;\, \mathsf{MF}^{\mathsf{M}}q \,;\, \mu_Z$$
$$= \quad \{\text{Definition of } ;_{\mathsf{M}}\}$$
$$\mathsf{F}^{\mathsf{M}}p \,;_{\mathsf{M}}\, \mathsf{F}^{\mathsf{M}}q$$

□

Claim. Lemma 6.11 stated that ϕ^{N} satisfies conditions (6.4), (6.5) and (6.6).

Proof. By the definition of the ordering in $\mathbf{Fun}(\mathsf{A})^{\mathsf{N}}$, ϕ^{N} satisfies the inequality (6.4) if we can show that for all functions $f : U \to X$ and $g : V \to Y$

$$(\mathsf{N}f \times \mathsf{N}g)\,;\phi^{\mathsf{N}}_{(X,Y)}\,;e_{X\times Y}\,;\ni \quad\subseteq\quad \phi^{\mathsf{N}}_{(U,V)}\,;\mathsf{N}(f\times g)\,;e_{X\times Y}\,;\ni$$

First, observe from the definition of ϕ^{N}, Definition 5.6, and equation (S.1) that

$$\phi^{\mathsf{N}}_{(X,Y)}\,;e_{X\times Y}\,;\ni \quad=\quad (e_X \times e_Y)\,;(\ni\times\ni)\,;((\in\times\in)\backslash\in) \tag{S.25}$$

We now calculate as follows. Note that the fact that $\times$ is a bifunctor will be used tacitly in the following proof and subscripts will be omitted:

$$\begin{array}{ll}
 & (\mathsf{N}f\times\mathsf{N}g)\,;\phi^{\mathsf{N}}\,;e\,;\ni \\
= & \{\text{Action of } \mathsf{N} \text{ on arrows (Definition 5.7), and equation (S.25)}\} \\
 & ((e\star\widehat{f})'\times(e\star\widehat{g})')\,;(e\times e)\,;(\ni\times\ni)\,;((\in\times\in)\backslash\in) \\
= & \{\text{Universal property from Definition 5.6}\} \\
 & ((e\star\widehat{f})\times(e\star\widehat{g}))\,;(\ni\times\ni)\,;((\in\times\in)\backslash\in) \\
\subseteq & \{\text{Property (S.20)}\} \\
 & (e\times e)\,;(\ni\times\ni)\,;((\in\times\in)\backslash(\widehat{f}\,;\ni\times\widehat{g}\,;\ni))\,;((\in\times\in)\backslash\in) \\
= & \{\text{Equation (4.6)}\} \\
 & (e\times e)\,;(\ni\times\ni)\,;((\in\times\in)\backslash((f\times g)\,;(\in\times\in)))\,;((\in\times\in)\backslash\in) \\
\subseteq & \{\text{Property (S.14)}\} \\
 & (e\times e)\,;(\ni\times\ni)\,;((\in\times\in)\backslash((f\times g)\,;\in)) \\
= & \{\text{Equation (4.6)}\} \\
 & (e\times e)\,;(\ni\times\ni)\,;((\in\times\in)\backslash(\widehat{f\times g}\,;\ni)) \\
= & \{\text{Equation (S.15)}\} \\
 & (e\times e)\,;(\ni\times\ni)\,;((\in\times\in)\backslash\in)\,;(\in\backslash(\widehat{f\times g}\,;\ni)) \\
= & \{\text{Equation (S.25)}\} \\
 & \phi^{\mathsf{N}}\,;e\,;\ni\,;(\in\backslash(\widehat{f\times g}\,;\ni)) \\
= & \{\text{Equation (S.19)}\} \\
 & \phi^{\mathsf{N}}\,;(e\star(\widehat{f\times g}))\,;\ni \\
= & \{\text{Action of } \mathsf{N} \text{ on arrows (Definition 5.7) }\} \\
 & \phi^{\mathsf{N}}\,;\mathsf{N}(f\times g))\,;e\,;\ni
\end{array}$$

The next step is to show that ϕ^{N} satisfies condition (6.5), which by the equivalence (A.2) is the same as showing that

$$(\eta_X\times\eta_Y)\,;\phi^{\mathsf{N}}_{(X,Y)}\,;e_{X\times Y} \quad=\quad \eta_{X\times Y}\,;e_{X\times Y} \tag{S.26}$$

The calculation is as follows:

$$
\begin{array}{ll}
& (\eta \times \eta)\,;\phi^{\mathsf{N}}\,;e \\
= & \{\text{Definitions 5.9 (of } \eta\text{) and 6.10 (of } \phi^{\mathsf{N}}\text{); universal property in Definition 5.6}\} \\
& (\iota \times \iota)\,;\Lambda((\ni \times \ni)\,;((\in \times \in)\backslash \in)) \\
= & \{(\text{S.3})\} \\
& \Lambda((\iota \times \iota)\,;(\ni \times \ni)\,;((\in \times \in)\backslash \in)) \\
= & \{\text{Definition 4.2 (of } \iota\text{); (S.1)}\} \\
& \Lambda((\in \times \in)\,;((\in \times \in)\backslash \in)) \\
= & \{(\text{S.22})\} \\
& \Lambda(\in) \\
= & \{\text{Definition of } \eta \text{ (see Definition 5.9)}\} \\
& \eta\,;e
\end{array}
$$

It remains to establish the last law, (6.6), which, by the definition of $\preceq_{\mathsf{N}}$ (see Theorem 5.10) is the same as

$$(\mu_X \times \mu_Y)\,;\phi^{\mathsf{N}}_{(X,Y)}\,;e_{X\times Y}\,;\ni \;\;\subseteq\;\; (\phi^{\mathsf{N}}_{(\mathsf{N}X,\mathsf{N}Y)}\,;_{\mathsf{N}}\,\phi^{\mathsf{N}}_{(X,Y)})\,;e_{X\times Y}\,;\ni$$

or, by (S.21) equivalently

$$(\mu_X \times \mu_Y)\,;\phi^{\mathsf{N}}_{(X,Y)}\,;e_{X\times Y}\,;\ni \;\;\subseteq\;\; ((\phi^{\mathsf{N}}_{(\mathsf{N}X,\mathsf{N}Y)}\,;e_{\mathsf{N}X\times\mathsf{N}Y})\star(\phi^{\mathsf{N}}_{(X,Y)}\,;e_{X\times Y}))\,;\ni$$

$$
\begin{array}{ll}
& (\mu \times \mu)\,;\phi^{\mathsf{N}}\,;e\,;\ni \\
= & \{\text{Definition of } \eta \text{ (see Definition 5.9); (S.25)}\} \\
& ((e \star e) \times (e \star e))\,;(\ni \times \ni)\,;((\in \times \in)\backslash \in) \\
\subseteq & \{(\text{S.20})\ \} \\
& (e \times e)\,;(\ni \times \ni)\,;((\in \times \in)\backslash((e \times e)\,;(\ni \times \ni))\,;((\in \times \in)\backslash \in) \\
\subseteq & \{(\text{S.18}))\ \} \\
& (e \times e)\,;(\ni \times \ni)\,;((\in \times \in)\backslash((e \times e)\,;(\ni \times \ni)\,;((\in \times \in)\backslash \in)) \\
= & \{(\text{S.25}))\ \} \\
& (e \times e)\,;(\ni \times \ni)\,;((\in \times \in)\backslash(\phi^{\mathsf{N}}\,;e\,;\ni)) \\
= & \{(\text{S.15}))\ \} \\
& (e \times e)\,;(\ni \times \ni)\,;((\in \times \in)\backslash \in)\,;(\in \backslash(\phi^{\mathsf{N}}\,;e\,;\ni)) \\
= & \{(\text{S.25})\ \} \\
& \phi^{\mathsf{N}}\,;e\,;\ni\,;(\in \backslash(\phi^{\mathsf{N}}\,;e\,;\ni)) \\
= & \{(\text{S.19})\ \} \\
& ((\phi^{\mathsf{N}}\,;e) \star (\phi^{\mathsf{N}}\,;e))\,;\ni
\end{array}
$$

□

Claim. Lemma 6.12 (the monadic multirelational fold)

Proof. Firstly, note that the following composition laws can be straightforwardly deduced from the definition of $;_{\mathsf{M}}$ and properties of M:

$$f ;_{\mathsf{M}} (g ; h) = (f ; \mathsf{M}g) ;_{\mathsf{M}} h \tag{S.27}$$
$$(f ; g) ;_{\mathsf{M}} h = f ; (g ;_{\mathsf{M}} h) \tag{S.28}$$

Then, let $\mathsf{F} : \mathsf{C} \to \mathsf{C}$ have initial algebra $\alpha : \mathsf{F}T \to T$. Suppose that $\mathsf{F}^{\mathsf{M}} : \mathsf{C}^{\mathsf{M}} \to \mathsf{C}^{\mathsf{M}}$ is a lax lifting, and let $\psi : \mathsf{F}X \to \mathsf{M}X$, $\tau : T \to \mathsf{M}X$ in C, then by Definition 6.2, $\tau = (\![\psi]\!)_{\mathsf{M}}$ if and only if

$$
\begin{aligned}
& (\alpha ; \eta) ;_{\mathsf{M}} \tau = \mathsf{F}^{\mathsf{M}}\tau ;_{\mathsf{M}} \psi \\
\equiv \quad & \{\text{Definition of } \mathsf{F}^{\mathsf{M}} \text{ (see Definition 6.8); (S.28)}\} \\
& \alpha ; \eta ;_{\mathsf{M}} \tau = \mathsf{F}\tau ; \delta^{\mathsf{F}} ;_{\mathsf{M}} \psi \\
\equiv \quad & \{\text{Identity of } ;_{\mathsf{M}}\} \\
& \alpha ; \tau = \mathsf{F}\tau ; \delta^{\mathsf{F}} ;_{\mathsf{M}} \psi \\
\equiv \quad & \{\text{Definition 6.2}\} \\
& \tau = (\![\delta^{\mathsf{F}} ;_{\mathsf{M}} \psi]\!)
\end{aligned}
$$

□

Claim. Lemma 6.13 (relationship of maps to folds)

Proof. Since $\alpha_X : \mathsf{F}(X, \mathsf{T}X) \to \mathsf{T}X$ is the initial algebra of bifunctor F considered as a unary functor F_X, where $\mathsf{F}_X(Y) = \mathsf{F}(X, Y)$ and $\mathsf{F}_X(f) = \mathsf{F}(id, f)$, Definition 6.2 implies that for all $p : \mathsf{F}(X, Y) \to Y$, $q : \mathsf{T}X \to Y$,

$$(\alpha ; q = \mathsf{F}(id, q) ; p) \quad \equiv \quad q = (\![p]\!) \tag{S.29}$$

and from this follows the fusion law:

$$h ; q = \mathsf{F}(id, q) ; p \quad \Rightarrow \quad (\![h]\!) ; q = (\![p]\!) \tag{S.30}$$

By (6.1) and Definition 6.8,

$$\mathsf{T}^{\mathsf{M}} f \;=\; \mathsf{T}f ; \delta^{\mathsf{T}} \;=\; (\![\mathsf{F}(f, id) ; \alpha]\!) ; \delta^{\mathsf{T}} \tag{S.31}$$

So we can calculate

$$
\begin{aligned}
& \mathsf{T}^{\mathsf{M}} f = (\![\mathsf{F}(f, id) ; \delta^{\mathsf{F}} ; \mathsf{M}\alpha]\!) \\
\equiv \quad & \{(\text{S.31})\} \\
& (\![\mathsf{F}(f, id) ; \alpha]\!) ; \delta^{\mathsf{T}} = (\![\mathsf{F}(f, id) ; \delta^{\mathsf{F}} ; \mathsf{M}\alpha]\!) \\
\Leftarrow \quad & \{(\text{S.30})\ \} \\
& \mathsf{F}(f, id) ; \alpha ; \delta^{\mathsf{T}} = \mathsf{F}(id, \delta^{\mathsf{T}}) ; \mathsf{F}(f, id) ; \delta^{\mathsf{F}} F ; \mathsf{M}\alpha \\
\equiv \quad & \{\mathsf{F} \text{ is a bifunctor }\}
\end{aligned}
$$

$$
\begin{aligned}
& \mathsf{F}(f, id)\,;\alpha\,;\delta^{\mathsf{T}} = \mathsf{F}(f, id)\,;\mathsf{F}(id, \delta^{\mathsf{T}})\,;\delta^{\mathsf{F}}\,;\mathsf{M}\alpha \\
\Leftarrow \quad & \{\text{Monotonicity of } ;\ \} \\
& \alpha\,;\delta^{\mathsf{T}} = \mathsf{F}(id, \delta^{\mathsf{T}})\,;\delta^{\mathsf{F}}\,;\mathsf{M}\alpha \\
\Leftarrow \quad & \{\text{Definition of } \delta^{\mathsf{T}} \text{ (see Definition 6.7)}\} \\
& \alpha\,;(\!|\delta^{\mathsf{F}}\,;\mathsf{M}\alpha|\!) = \mathsf{F}(id, \delta^{\mathsf{T}})\,;\delta^{\mathsf{F}}\,;\mathsf{M}\alpha \\
\equiv \quad & \{\text{(S.29)}\} \\
& True
\end{aligned}
$$

First note that if the family of arrows δ^{F} satisfies condition (6.4), then it can be used to define a family δ^{F_X}, where

$$\delta^{\mathsf{F}_X} \ \widehat{=}\ \mathsf{F}(\eta, id)\,;\delta^{\mathsf{F}} \qquad \text{(S.32)}$$

Finally, the following proof is just like that in [14] except that some equalities have been replaced by inequalities. Note that f has type $f : Y \to \mathsf{M}Z$, for some objects Y and Z.

$$
\begin{aligned}
& (\!|\mathsf{F}^{\mathsf{M}}(f, \eta)\,;_{\mathsf{M}}(\alpha\,;\eta)|\!)_{\mathsf{M}} \\
= \quad & \{\text{Definition of } \mathsf{F}^{\mathsf{M}} \text{ and (S.28)}\} \\
& (\!|\mathsf{F}(f, \eta)\,;\delta^{\mathsf{F}}\,;_{\mathsf{M}}(\alpha\,;\eta)|\!)_{\mathsf{M}} \\
= \quad & \{\text{Lemma 6.12 (monadic catamorphisms)}\} \\
& (\!|\delta^{\mathsf{F}_X}\,;_{\mathsf{M}}(\mathsf{F}(f, \eta)\,;\delta^{\mathsf{F}}\,;_{\mathsf{M}}(\alpha\,;\eta)))|\!) \\
= \quad & \{\ \text{(S.32)}\} \\
& (\!|\mathsf{F}(\eta, id)\,;\delta^{\mathsf{F}}\,;_{\mathsf{M}}(\mathsf{F}(f, \eta)\,;\delta^{\mathsf{F}}\,;_{\mathsf{M}}(\alpha\,;\eta)))|\!) \\
= \quad & \{\ \text{(S.27)}\} \\
& (\!|(\mathsf{F}(\eta, id)\,;\delta^{\mathsf{F}}\,;\mathsf{MF}(f, \eta))\,;_{\mathsf{M}}\delta^{\mathsf{F}}\,;_{\mathsf{M}}(\alpha\,;\eta))|\!) \\
\succeq \quad & \{\ \text{(6.4)}\} \\
& (\!|(\mathsf{F}(\eta, id)\,;\mathsf{F}(\mathsf{M}f, \mathsf{M}\eta))\,;\delta^{\mathsf{F}}\,;_{\mathsf{M}}\delta^{\mathsf{F}}\,;_{\mathsf{M}}(\alpha\,;\eta))|\!) \\
= \quad & \{\mathsf{F} \text{ is a functor; naturality of } \eta\text{; type of } f\} \\
& (\!|(\mathsf{F}(f, id)\,;\mathsf{F}(\eta\mathsf{M}, \mathsf{M}\eta))\,;(\delta^{\mathsf{F}}\,;_{\mathsf{M}}\delta^{\mathsf{F}})\,;_{\mathsf{M}}(\alpha\,;\eta))|\!) \\
= \quad & \{\text{(S.27); properties of } ;_{\mathsf{M}} \text{ then (S.28)}\} \\
& (\!|\mathsf{F}(f, id)\,;\mathsf{F}(\eta\mathsf{M}, \mathsf{M}\eta)\,;(\delta^{\mathsf{F}}\,;_{\mathsf{M}}\delta^{\mathsf{F}})\,;\mathsf{M}\alpha|\!) \\
\succeq \quad & \{\ \text{(6.6)}\ \} \\
& (\!|\mathsf{F}(f, id)\,;\mathsf{F}(\eta\mathsf{M}, \mathsf{M}\eta)\,;\mathsf{F}(\mu, \mu)\,;\delta^{\mathsf{F}}\,;\mathsf{M}\alpha|\!) \\
= \quad & \{\mathsf{F} \text{ is a functor, monad properties }\} \\
& (\!|\mathsf{F}(f, id)\,;\delta^{\mathsf{F}}\,;\mathsf{M}\alpha|\!)
\end{aligned}
$$

□

Proofs of Laws in this Supplement

Finally, we address the laws listed earlier in this supplement.

Laws S.1 and S.2 are immediate from the universal property (2.1). □

Proof. (S.3)

$$
\begin{aligned}
& f\,;\Lambda r = \Lambda(f\,;r) \\
\equiv\quad & \{\text{Universal property in Definition 2.1}\} \\
& f\,;\Lambda r\,;\ni\ = f\,;r \\
\equiv\quad & \{(\text{S.1})\} \\
& f\,;r = f\,;r
\end{aligned}
$$

□

Proof. (S.4)

$$
\begin{aligned}
& \mathsf{E}r\,;\ni\ =\ \ni;r \\
\equiv\quad & \{\text{Universal property in Definition 2.1}\} \\
& \mathsf{E}r = \Lambda(\ni;r) \\
\equiv\quad & \{\text{Definition of } \mathsf{E}\} \\
& True
\end{aligned}
$$

□

Proof. (S.5)

$$
\begin{aligned}
& \Lambda(r\,;s) = \Lambda r\,;\mathsf{E}s \\
\equiv\quad & \{\text{Universal property in Definition 2.1}\} \\
& r\,;s = \Lambda r\,;\mathsf{E}s\,;\ni \\
\equiv\quad & \{(\text{S.4})\} \\
& r\,;s = \Lambda r\,;\ni;s \\
\equiv\quad & \{(\text{S.1})\} \\
& r\,;s = r\,;s
\end{aligned}
$$

□

Proof. (S.6)

$$
\begin{array}{ll}
 & \Lambda(\Lambda^{-1} f) \\
= & \{\text{Definition of } \Lambda^{-1}\} \\
 & \Lambda(f\,;\ni) \\
\equiv & \{(\text{S.2})\} \\
 & f
\end{array}
$$

□

Proof. (S.7)

$$
\begin{array}{ll}
 & \Lambda^{-1}(\Lambda r) \\
= & \{\text{Definition of } \Lambda^{-1}\} \\
 & \Lambda r\,;\ni \\
\equiv & \{(\text{S.1})\} \\
 & r
\end{array}
$$

□

The hint *shunting* in the following proofs refers to use of one of the shunting rules (e.g. see [12]) for functions:

$$
\begin{array}{rcl}
f^\circ\,;r \subseteq s & \equiv & r \subseteq f\,;s \\
r\,;f \subseteq s & \equiv & r \subseteq s\,;f^\circ
\end{array}
$$

Proof. (S.8)

$$
\begin{array}{ll}
 & t \subseteq (f^\circ\,;r)\backslash s \\
= & \{\text{Definition 2.5 (of } \backslash)\} \\
 & f^\circ\,;r\,;t \subseteq s \\
= & \{\text{Shunting}\} \\
 & r\,;t \subseteq f\,;s \\
= & \{\text{Definition 2.5}\} \\
 & t \subseteq r\backslash(f\,;s)
\end{array}
$$

□

Proof. (S.9)

$$
\begin{array}{ll}
 & t \subseteq f\,;(r\backslash s) \\
= & \{\text{Shunting}\}
\end{array}
$$

$$
\begin{array}{ll}
 & f^\circ ; t \subseteq r\backslash s \\
= & \quad \{\text{Definition 2.5 (of } \backslash)\} \\
 & r ; f^\circ ; t \subseteq s \\
= & \quad \{\text{Definition 2.5}\} \\
 & t \subseteq (r ; f^\circ)\backslash s
\end{array}
$$

□

Proof. (S.10)

$$
\begin{array}{ll}
 & \mathsf{E}f ; (\in \backslash r) \\
= & \quad \{\text{S.9}\} \\
 & (\in ; (\mathsf{E}f)^\circ)\backslash r \\
= & \quad \{\text{Converse of (S.4)}\} \\
 & (f^\circ ; \in)\backslash r \\
= & \quad \{\text{(S.8}\} \\
 & \in \backslash (f ; r)
\end{array}
$$

□

The proofs of laws (S.11), (S.12) and (S.13) are omitted since they dual to those given for / in Sections 2.314 and 2.31 of [17]. □

Proof. (S.14)

$$
\begin{array}{ll}
 & (r\backslash(s ; t)) ; (t\backslash u) \subseteq r\backslash(s ; u) \\
\equiv & \quad \{\text{Definition 2.5 (of } \backslash)\} \\
 & r ; (r\backslash(s ; t)) ; (t\backslash u) \subseteq s ; u \\
\Leftarrow & \quad \{\text{(S.13)}\} \\
 & s ; t ; (t\backslash u) \subseteq s ; u \\
\Leftarrow & \quad \{\text{(S.13)}\} \\
 & s ; u \subseteq s ; u
\end{array}
$$

□

Law (S.15) is proved in [18]. □

Proof. (S.16)

$$
\begin{array}{ll}
& (r\backslash s) \times (t\backslash u) \subseteq (r \times t)\backslash(s \times u) \\
\equiv & \{\text{Definition 2.5 (of } \backslash)\} \\
& (r \times t)\,;((r\backslash s) \times (t\backslash u)) \subseteq s \times u \\
\equiv & \{\times \text{ is a bifunctor}\} \\
& (r\,;(r\backslash s)) \times (t\,;(t\backslash u)) \subseteq s \times u \\
\equiv & \{\text{(S.13) and } \times \text{ is monotonic}\} \\
& \mathit{True}
\end{array}
$$

□

The proof of (S.17) follows the same outline as (S.10). □

Proof. (S.18)

$$
\begin{array}{ll}
& (r\backslash s)\,;t \subseteq r\backslash(s\,;t) \\
\equiv & \{\text{Definition 2.5 (of } \backslash)\} \\
& r\,;(r\backslash s)\,;t \subseteq s\,;t \\
\equiv & \{\text{(S.13)}\} \\
& \mathit{True}
\end{array}
$$

□

Laws (S.19) and (S.21) are just a restatement of the previous observation that each isomorphism of categories distributes through composition.

Proof. (S.20)

$$
\begin{array}{ll}
& (f \star g \;\times\; h \star k)\,;(\ni \times \ni) \\
\subseteq & \{\times \text{ is a bifunctor}\} \\
& ((f \star g)\,;\ni) \;\times\; ((h \star k)\,;\ni) \\
\subseteq & \{\text{(S.19)}\} \\
& (f\,;\ni;(\in \backslash(g\,;\ni))) \;\times\; (h\,;\ni;(\in \backslash(k\,;\ni))) \\
\subseteq & \{\times \text{ is a bifunctor}\} \\
& (f \times h)\,;(\ni \times \ni)\,;((\in \backslash(g\,;\ni))) \;\times\; (\in \backslash(k\,;\ni))) \\
\subseteq & \{\text{(S.16)}\} \\
& (f \times h)\,;(\ni \times \ni)\,;((\in \times \in)\backslash((g \times k)\,;(\ni \times \ni)))
\end{array}
$$

□

Proof. (S.22) First note that if there exists a function p such that $id = r\,;p^{\circ}$, then $id \subseteq r\,;p^{\circ}$ and by the second shunting rule we have $p \subseteq r$. By (S.13), $r\,;(r\backslash t) \subseteq t$, and so it is sufficient to establish the reverse inclusion: $t \subseteq r\,;(r\backslash t)$:

$$
\begin{array}{ll}
 & t \\
= & \{(\text{S.12})\} \\
 & id\backslash t \\
= & \{\text{Assumption}\} \\
 & (r\,;p^{\circ})\backslash t \\
= & \{(\text{S.9})\} \\
 & p\,;(r\backslash t) \\
\subseteq & \{p \subseteq r \text{ and monotonicity of } ;\} \\
 & r\,;(r\backslash t)
\end{array}
$$

□

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[12] R.S. Bird and O. de Moor. *The Algebra of Programming.* Prentice Hall, 1997.

[17] Peter Freyd and Andre Ščedrov. *Categories, Allegories.* Springer Verlag, 1993.

[18] Martin, C.E., Curtis, S.A.: The algebra of multirelations (2009) (in preparation).