Semantics of the relative belief of singletons

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Summary. In this paper we introduce the relative belief of singletons as a novel Bayesian approximation of a belief function. We discuss its nature in terms of degrees of belief under several different angles, and its applicability to different classes of belief functions.

Key words: Theory of evidence, Bayesian approximation, relative plausibility and belief of singletons.

1 Introduction: A new Bayesian approximation

The theory of evidence (ToE) [16] extends classical probability theory through the notion of *belief function* (b.f.), a mathematical entity which independently assigns probability values to *sets* of possibilities rather than single events. A belief function $b : 2^\Theta \rightarrow [0,1]$ on a finite set ("frame") Θ has the form $b(A) = \sum_{B \subseteq A} m_b(B)$ where $m_b : 2^\Theta \rightarrow [0,1]$, is called "basic probability assignment" (b.p.a.), and meets the positivity $m_b(A) \geq 0 \ \forall A \subseteq \Theta$ and normalization $\sum_{A \subseteq \Theta} m_b(A) = 1$ axioms. Events associated with non-zero basic probabilities are called "focal elements". A b.p.a. can be uniquely recovered from a belief function through Moebius inversion:

$$m_b(A) = \sum_{B \subseteq A} (-1)^{|A-B|} b(B). \tag{1}$$

1.1 Previous work on Bayesian approximation

As probability measures or *Bayesian* belief functions are just a special class of b.f.s (for which $m(A) = 0$ when $|A| > 1$), the relation between beliefs and probabilities plays a major role in the theory of evidence [9, 14, 23, 11, 12, 13, 2]. Tessem [21], for instance, incorporated only the highest-valued focal elements in his m_{klx} approximation; a similar approach inspired the summarization technique formulated by Lowrance *et al.* [15]. In Smets' "Transferable Belief Model" [17] beliefs are represented at credal level (as convex sets of probabilities), while decisions are made by resorting to a Bayesian belief function called pignistic transformation [19]. More recently, two new Bayesian approximations of belief functions have been derived from purely geometric considerations [7] in the context of the geometric approach to the ToE [6], in which belief and probability measures are represented as points of a Cartesian space.

Another classical approximation is based on the plausibility function (pl.f.) $pl_b : 2^{\Theta} \rightarrow [0, 1]$, where

$$pl_b(A) \doteq 1 - b(A^c) = \sum_{B \cap A \neq \emptyset} m_b(B)$$

represent of the evidence not against a proposition A.
Voorbraak [22] proposed indeed to adopt the so-called *relative plausibility of singletons* (rel.plaus.) $\tilde{pl}_b$ as the unique probability that, given a belief function b with plausibility pl_b, assigns to each singleton $x \in \Theta$ its normalized plausibility (2). He proved that $\tilde{pl}_b$ is a perfect representative of b when combined with other probabilities p through Dempster's rule $\oplus$ [10],

$$\tilde{pl}_b(x) = \frac{pl_b(x)}{\sum_{y \in \Theta} pl_b(y)}, \qquad \tilde{pl}_b \oplus p = b \oplus p. \tag{2}$$

The properties of the relative plausibility of singletons have been later discussed by Cobb and Shenoy [3].

1.2 Relative belief of singletons

In this paper we introduce indeed a Bayesian approximation which is the *dual* of relative plausibility of singletons (2), as it is obtained by normalizing the *belief* (instead of plausibility) values of singletons:

$$\tilde{b}(x) \doteq \frac{b(x)}{\sum_{y \in \Theta} b(y)} = \frac{m_b(x)}{\sum_{y \in \Theta} m_b(y)}. \tag{3}$$

We call it *relative belief of singletons* $\tilde{b}$ (rel.bel.). Clearly $\tilde{b}$ exists iff b assigns some mass to singletons:

$$\sum_{x \in \Theta} m_b(x) \neq 0. \tag{4}$$

As it has been recently proven [4], both relative plausibility and belief of singletons commute with respect to Dempster's orthogonal sum, and $\tilde{b}$ meets the dual of Voorbraak's representation theorem (2).

Proposition 1. *The relative belief operator commutes with respect to Dempster's combination of plausibility functions, namely*

$$\tilde{b}[pl_1 \oplus pl_2] = \tilde{b}[pl_1] \oplus \tilde{b}[pl_2].$$

The relative belief of singletons $\tilde{b}$ *represents perfectly the corresponding plausibility function* pl_b *when combined with any probability through (extended) Dempster's rule:*

$$\tilde{b} \oplus p = pl_b \oplus p$$

for each Bayesian belief function $p \in \mathcal{P}$*.*

Moreover, $\tilde{b}$ meets a number of properties with respect to Dempster's rule which mirror the set of results proven by Cobb and Shenoy for the relative plausibility of singletons [3].

Proposition 2. *If* pl_b *is idempotent with respect to Dempster's rule, i.e.* $pl_b \oplus pl_b = pl_b$*, then* $\tilde{b}[pl_b]$ *is itself idempotent:* $\tilde{b}[pl_b] \oplus \tilde{b}[pl_b] = \tilde{b}[pl_b]$*.*
If $\exists x \in \Theta$ *such that* $b(x) > b(y)$ $\forall y \neq x, y \in \Theta$*, then* $\tilde{b}[pl_b^\infty](x) = 1$*,* $\tilde{b}[pl_b^\infty](y) = 0$ $\forall y \neq x$*, where* pl_b^∞ *denotes the infinite limit of the combination of* pl_b *with itself.*

In this paper we focus instead on the *semantics* of rel.bel. in a comparative study with that of rel.plaus., in order to understand its meaning in terms of degrees of belief, the way it attributes a mass to singletons, the conditions under which it exists, and to which classes of belief function it can be applied.

1.3 Outline of the paper

First (Section 2), we argue that rel.bel. gives a conservative estimate of the support b give to each singleton $x \in \Theta$, in opposition to the optimistic estimate provided by the relative plausibility of singletons. Interestingly (Section 3) the relative belief $\tilde{b}$ can indeed be interpreted as the relative plausibility of singletons *of the associated plausibility function.* In order to prove that, we need to extend the evidential formalism to functions whose Moebius inverse is not necessarily positive or *pseudo belief functions* (Section 3.1). Those two Bayesian approximations form then a couple which, besides having dual properties with respect to Dempster's sum, have dual semantics in terms of mass assignment.
In Section 4 we analyze the issue posed by the existence constraint (4), i.e. the fact that rel.bel. exists only when b assigns some mass to singletons. We will argue that situations in which the latter is not met are pathological, as all Bayesian approximations are forced to span a limited region of the probability simplex. Finally, we will prove that, as all those approximations converge for quasi-Bayesian b.f.s, rel.bel. can be seen as a low-cost proxy to pignistic transformation and relative plausibility, and discuss the applicability of $\tilde{b}$ to some important classes of b.f.s in order to shed more light on interpretation and application range of this Bayesian approximation.

2 A conservative estimate

A first insight on the meaning of $\tilde{b}$ comes from the original semantics of belief functions as constraints on the actual allocation of mass of an underlying unknown probability distribution. Accordingly, a focal element A with mass $m_b(A)$ indicates that this mass can "float" around in A and be distributed arbitrarily between the elements of A. In this framework, the relative plausibility of singletons $\tilde{pl}_b$ (2) can be interpreted as follows:

- for each singleton $x \in \Theta$ the most *optimistic* hypothesis in which the mass of all $A \supseteq \{x\}$ focuses on x is considered, yielding $\{pl_b(x), x \in \Theta\}$;
- this assumption, however, is contradictory as it is supposed to hold for all singletons (many of which belong to the same higher-size events);
- nevertheless, the obtained values are normalized to yield a Bayesian belief function.

$\tilde{pl}_b$ is associated with the less conservative (but incoherent) scenario in which all the mass that can be assigned to a singleton is actually assigned to it.
The relative belief of singletons (3) can then be naturally given the following interpretation in terms of mass assignments:

- for each singleton $x \in \Theta$ the most *pessimistic* hypothesis in which only the mass of $\{x\}$ itself actually focuses on x is considered, yielding $\{b(x) = m_b(x), x \in \Theta\}$;
- this assumption is also contradictory, as the mass of all higher-size events is not assigned to any singletons;
- the obtained values are again normalized to produce a Bayesian belief function.

Dually, $\tilde{b}$ reflects the most conservative (but still not coherent) choice of assigning to x only the mass that the b.f. b (seen as a constraint) assures it belong to x. The underlying mechanism, though, is exactly the same as the one supporting the rel.plaus. function.

3 Dual interpretation as relative plausibility of a plausibility

A different aspect of rel.bel. emerges when considering the dual representation of the evidence carried by b expressed by the plausibility function pl_b. We first need though to introduce the notion of "pseudo belief function".

3.1 Pseudo belief functions

A belief function is a function on 2^Θ whose Moebius inverse m_b (the basic probability assignment) meets the positivity axiom: $m_b(A) \geq 0 \ \forall A \subseteq \Theta$.

However, all functions $\varsigma : 2^\Theta \rightarrow \mathbb{R}$ admit Moebius inverse (1) $m_\varsigma : 2^\Theta \setminus \emptyset \rightarrow \mathbb{R}$ such that

$$\varsigma(A) = \sum_{B \subseteq A} m_\varsigma(B)$$

where $m_\varsigma(B) \not\geq 0 \ \forall B \subseteq \Theta$ [1].
Functions ς whose Moebius inverse meets the normalization constraint

$$\sum_{\emptyset \subsetneq A \subseteq \Theta} m_\varsigma(A) = 1$$

are then natural extensions of belief functions[1], and are called *pseudo belief functions* (p.b.f.s) [20].
As they meet the normalization constraint ($pl_b(\Theta) = 1$ for all b), plausibility functions are themselves pseudo belief functions. Their Moebius inverse [8]

$$\mu_b(A) \doteq \sum_{B \subseteq A} (-1)^{|A \setminus B|} pl_b(B) = (-1)^{|A|+1} \sum_{B \supseteq A} m_b(B), \ \ A \neq \emptyset \qquad (5)$$

is called *basic plausibility assignment* (b.pl.a.), with $\mu_b(\emptyset) = 0$.

3.2 Duality between relative belief and plausibility

A useful property of μ_b is that

Theorem 1. *$m_b(x) = \sum_{A \supseteq \{x\}} \mu_b(A)$.*

If we write the plausibility of singletons as

$$pl_b(x) = \sum_{A \supseteq \{x\}} m_b(A)$$

we realize that Theorem 1 states that the belief of singletons $b(x)$ is nothing but *the plausibility of singletons of pl_b interpreted as a pseudo belief function*: $b(x) = pl_{pl_b}(x)$. Formally,

- there exists a class of pseudo b.f.s which correspond to the plausibility of some b.f. b: $\varsigma = pl_b$ for some $b \in \mathcal{B}$;
- each p.b.f. admits a (pseudo) plausibility function, analogous to the case of standard b.f.s: $pl_\varsigma(A) = \sum_{B \cap A \neq \emptyset} m_\varsigma(B)$;
- but for the above class of p.b.f. $\varsigma = pl_b$, so that the above equation reads as $pl_{pl_b}(A) = \sum_{B \cap A \neq \emptyset} \mu_b(B)$ (as μ_b is the Moebius inverse of pl_b);
- when applied to singletons this yields

$$pl_{pl_b}(x) = \sum_{B \ni x} \mu_b(B) = m_b(x) \qquad (6)$$

  by Theorem 1, which implies $\tilde{pl}_{pl_b} = \tilde{b}$.

[1] Geometrically, each p.b.f. can be thought of as a vector ς of $\mathbb{R}^N$, $N = 2^{|\Theta|} - 1$, while belief functions form a *simplex* in the same space [6].

It is a bit paradoxical to point out that, as the basic plausibility assignment μ_b carries the same information as the basic probability assignment m_b, according to Equation (6) *all* the information carried by b is used to compute the relative belief of singletons, while its definition (3) seems to suggest that most of this information is discarded in the process.

4 On the existence constraint

The relative belief of singletons exists only for those belief function such that $\sum_x b(x) \neq 0$.
As $\tilde{b}$ is the relative plausibility of $\varsigma = pl_b$ (Section 3), and as relative plausibilities do not undergo any existence constraint (as $\sum_x pl_b(x) \neq 0$), one could argue that $\tilde{b}$ should always exist. However, the symmetry is broken by the fact that the b.pl.a. μ_b does not meet the non-negativity constraint ($\mu_b \ngeq 0$), and as a consequence $pl_{pl_b}(x)$ can actually be zero $\forall x \in \Theta$.

4.1 Example: the binary case

In the binary case $\Theta = \{x, y\}$, for instance, according to (4) the only b.f. which does not admit rel.bel. is the vacuous one b_Θ: $m_{b_\Theta}(\Theta) = 1$. For b_Θ, $m_{b_\Theta}(x) = m_{b_\Theta}(y) = 0$ so that $\sum_x m_{b_\Theta}(x) = 0$ and $\tilde{b}_\Theta$ does not exist. Symmetrically, the pseudo b.f. $\varsigma = pl_{b_\Theta}$ (for which $pl_{b_\Theta}(x) = pl_{b_\Theta}(y) = 1$) is such that $pl_{pl_{b_\Theta}} = b_\Theta$, so that $\tilde{pl}_{pl_{b_\Theta}}$ does not exist. In the binary frame each belief

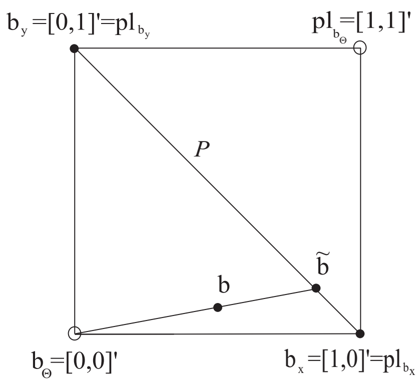

Fig. 1. B.f.s $b = [m_b(x), m_b(y)]'$ and pl.f.s $pl_b = [pl_b(x) = 1 - m_b(y), pl_b(y) = 1 - m_b(x)]'$ on $\Theta = \{x, y\}$ can be represented as points of $\mathbb{R}^2$ [6]. The locations of $\tilde{b} = [\frac{m_b(x)}{m_b(x)+m_b(y)}, \frac{m_b(y)}{m_b(x)+m_b(y)}]'$ and the singular points $b_\Theta = [0,0]'$ and $pl_{b_\Theta} = [1,1]'$ are shown.

function is completely determined by its belief values $b(x), b(y)$ (as $b(\emptyset) = 0$, $b(\Theta) = 1$ for all b) and can then be represented as a point of the plane $\mathbb{R}^2$:

$$b = [b(x), b(y)]'.$$

Figure 1 illustrates then the location of $\tilde{b}$ in the simple binary case and those of the dual singular points b_Θ, $\varsigma = pl_{b_\Theta}$.

4.2 Region spanned by a Bayesian approximation

One can argue that the existence of rel.bel. is subject to quite a strong condition (4). We can claim though that situations in which the constraint is not met are indeed rather pathological, in a very precise way.
To show this, let us compute the region spanned by the most common Bayesian approximations: rel.plaus. (2) and *pignistic function* [19]

$$BetP[b](x) \doteq \sum_{A \supseteq \{x\}} \frac{m_b(A)}{|A|}.$$

All Bayesian approximations can be seen as operators mapping belief functions to probabilities:

$$\begin{array}{rl} \tilde{pl} : \mathcal{B} \rightarrow & \mathcal{P} \\ b \mapsto & \tilde{pl}[b] = \tilde{pl}_b \end{array} \qquad \begin{array}{rl} BetP : \mathcal{B} \rightarrow & \mathcal{P} \\ b \mapsto & BetP[b] \end{array} \tag{7}$$

where $\mathcal{B}$ and $\mathcal{P}$ denote the set of all b.f.s and probability functions respectively.
Now, it is well known [7] that the pignistic transformation (7)-right commutes with affine combination:

$$BetP\Big[\sum_i \alpha_i b_i\Big] = \sum_i \alpha_i BetP[b_i], \qquad \sum_i \alpha_i = 1. \tag{8}$$

If we then denote by Cl the convex closure operator

$$Cl(b_1, ..., b_k) = \left\{ b \in \mathcal{B} : b = \alpha_1 b_1 + \cdots + \alpha_k b_k, \sum_i \alpha_i = 1,\ \alpha_i \geq 0\ \forall i \right\} \tag{9}$$

(8) implies that $BetP$ commutes with Cl:

$$BetP[Cl(b_1, ..., b_k)] = Cl(BetP[b_i], i = 1, ..., k).$$

In the case of $\tilde{pl}_b$, even though the latter does not commute with affine combination (the relation being somehow more complex [5]) we can still prove that it commutes with convex closure (9).
Using this tools we can find the region of the probability simplex $\mathcal{P}$ spanned by the Bayesian transformation of a certain convex region $Cl(b_1, ..., b_k)$ of b.f.s. It suffices to compute in both cases the approximations of the vertices of the considered region.

4.3 Zero mass to singletons as a pathological situation

But the space of all belief functions $\mathcal{B} \doteq \{b : 2^\Theta \rightarrow [0,1]\}$ defined on a frame Θ is indeed the convex closure [6]

$$\mathcal{B} = Cl(b_A, A \subseteq \Theta) \tag{10}$$

of all *basis* belief functions

$$b_A \doteq b \in \mathcal{B} \; s.t. \; m_b(A) = 1, \, m_b(B) = 0 \; \forall B \neq A \tag{11}$$

i.e. the belief functions focusing on a single event $A \subseteq \Theta$. Geometrically, they are the vertices of the polytope $\mathcal{B}$ of all belief functions (Figure 2-left).

The images of a basis b.f. b_A under the transformations (7) are

$$\begin{array}{l} \tilde{pl}_{b_A}(x) = \frac{\sum_{B \supseteq \{x\}} m_{b_A}(B)}{\sum_{B \supseteq \{x\}} m_{b_A}(B)|B|} = \begin{cases} \frac{1}{|A|} & x \in A \\ 0 & else \end{cases} \doteq \overline{\mathcal{P}}_A \\ BetP[b_A](x) = \sum_{B \supseteq \{x\}} \frac{m_{b_A}(B)}{|B|} = \overline{\mathcal{P}}_A \end{array}$$

so that

$$\begin{array}{c} BetP[\mathcal{B}] = BetP[Cl(b_A, A \subseteq \Theta)] = Cl(BetP[b_A], A \subseteq \Theta) = \\ = Cl(\overline{\mathcal{P}}_A, A \subseteq \Theta) = \mathcal{P} = \tilde{pl}[\mathcal{B}]. \end{array}$$

In normal conditions *the whole* probability simplex $\mathcal{P}$ can host such approx-

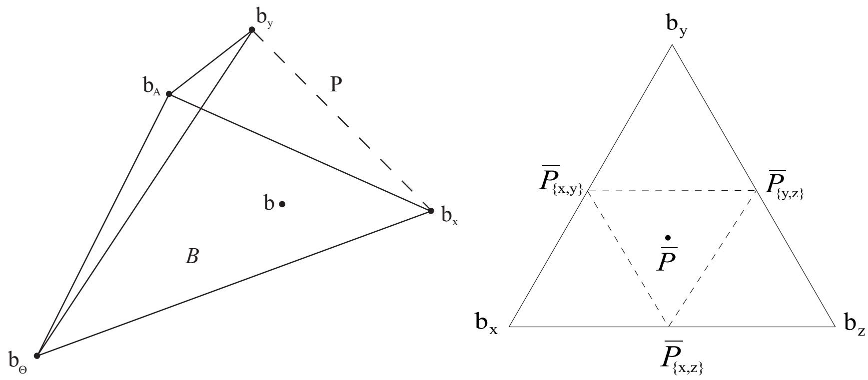

Fig. 2. Left: The space of all belief functions on a domain Θ is a polytope or "simplex" (10) in $\mathbb{R}^N$. The probability simplex $\mathcal{P}$ is a face of $\mathcal{B}$. Right: For the class of b.f.s $\{b : \sum_x m_b(x) = 0\}$, pignistic function and relative plausibility are allowed to span only a proper subset of the probability simplex (delimited by dashed lines in the ternary case $\Theta = \{x, y, z\}$). Otherwise $\tilde{b}$, $BetP[b]$, $\tilde{pl}_b$ can be located in any point of $\mathcal{P}$ for some values of b.

imations. On the other side, as they have the form $b = \sum_{|A|>1} m_b(A) b_A$ with

$m_b(A) \geq 0$, $\sum_{|A|>1} m_b(A) = 1$, the set of (singular) b.f.s *not* meeting the constraint (4) is $Cl(b_A, |A| > 1)$ so that the region of $\mathcal{P}$ spanned by their Bayesian approximations is

$$\tilde{pl}[Cl(b_A, |A| > 1)] = Cl(\tilde{pl}_{b_A}, |A| > 1) = Cl(\overline{\mathcal{P}}_A, |A| > 1) = \\ = Cl(BetP[b_A], |A| > 1) = BetP[Cl(b_A, |A| > 1)].$$

The result is illustrated by Figure 2-right in the ternary case. If (4) is not met, all Bayesian approximations of b can span only a limited region

$$Cl(\overline{\mathcal{P}}_{\{x,y\}}, \overline{\mathcal{P}}_{\{x,z\}}, \overline{\mathcal{P}}_{\{y,z\}}, \overline{\mathcal{P}}_{\Theta}) = Cl(\overline{\mathcal{P}}_{\{x,y\}}, \overline{\mathcal{P}}_{\{x,z\}}, \overline{\mathcal{P}}_{\{y,z\}})$$

of the probability simplex (delimited by dashed lines).
The case in which $\tilde{b}$ does not exist is indeed pathological, as it excludes a great deal of belief and probability measures.

5 A low-cost proxy for other Bayesian approximations

A different angle on the utility of $\tilde{b}$ comes from a discussion of what classes of b.f.s are "suitable" to be approximated by means of (3). As it only makes use of the masses of singletons, working with $\tilde{b}$ requires storing n values to represent a belief function. As a consequence, the computational cost of combining new evidence through Dempster's rule or disjunctive combination [18] is reduced to $O(n)$ as only the mass of singletons has to be calculated.
When the actual values of $\tilde{b}(x)$ are close to those provided by, for instance, pignistic function or rel.plaus. is then more convenient to resort to the relative belief transformation.

5.1 Convergence under quasi-Bayesianity

A formal support to this argument is provided by the following result. Let us call *quasi-Bayesian* b.f.s the belief functions b for which the mass assigned to singletons is very close to one:

$$k_{m_b} \doteq \sum_{x \in \Theta} m_b(x) \to 1.$$

Theorem 2. *For quasi-Bayesian b.f.s all Bayesian approximations converge:*

$$\lim_{k_{m_b} \to 1} BetP[b] = \lim_{k_{m_b} \to 1} \tilde{pl}_b = \lim_{k_{m_b} \to 1} \tilde{b}.$$

Theorem 2 highlights then the convenience of computing rel.bel. instead of other Bayesian approximations for quasi-Bayesian b.f.s defined on a large frame of discernment.

5.2 Convergence in the ternary case

Let us consider for instance the ternary case $\Theta = \{x, y, z\}$ in which

$$\tilde{b}(x) = \frac{m_b(x)}{m_b(x) + m_b(y) + m_b(z)},$$
$$\tilde{pl}_b(x) = \frac{(m_b(x) + m_b(\{x,y\}) + m_b(\{x,z\}) + m_b(\Theta))}{\sum_{w\in\Theta} pl_b(w)}, \qquad (12)$$
$$BetP[b](x) = m_b(x) + \frac{m_b(\{x,y\}) + m_b(\{x,z\})}{2} + \frac{m_b(\Theta)}{3}.$$

with $\sum_{w\in\Theta} pl_b(w) = (m_b(x) + m_b(y) + m_b(z)) + 2(m_b(\{x,y\}) + m_b(\{x,z\}) + m_b(\{y,z\})) + 3m_b(\Theta)$.
According to Theorem 2, if $k_{m_b} = \sum_w m_b(w) \to 1$ all the quantities (12) converge to $m_b(x)$. It is interesting to assess the velocity of this convergence with respect to the parameter k_{m_b}. For sake of comparison we consider two different mass allocations to higher-order events: a) $m_b(\Theta) = 1 - k_{m_b}$ and: b) $m_b(\{x,y\}) = 1 - k_{m_b}$. The above expressions (12) then yield

$$\tilde{b}(w) = \frac{m_b(w)}{k_{m_b}} \quad \forall w \in \{x,y,z\};$$
$$\tilde{pl}_b(w) = \begin{cases} \frac{m_b(w)+1-k_{m_b}}{k_{m_b}+3(1-k_{m_b})} \; \forall w \in \{x,y,z\} & m_b(\Theta) = 1 - k_{m_b}, \\ \frac{m_b(w)+1-k_{m_b}}{k_{m_b}+2(1-k_{m_b})} \; \forall w \in \{x,y\} & m_b(\{x,y\}) = 1 - k_{m_b}, \\ \frac{m_b(w)}{k_{m_b}+2(1-k_{m_b})} \; w = z & m_b(\{x,y\}) = 1 - k_{m_b}; \end{cases}$$
$$BetP[b](w) = \begin{cases} m_b(w) + \frac{1-k_{m_b}}{3} \; \forall w \in \{x,y,z\} & m_b(\Theta) = 1 - k_{m_b}, \\ m_b(w) + \frac{1-k_{m_b}}{2} \; \forall w \in \{x,y\} & m_b(\{x,y\}) = 1 - k_{m_b}, \\ m_b(w) \quad w = z & m_b(\{x,y\}) = 1 - k_{m_b}. \end{cases}$$

We evaluated the above expressions for $m_b(x) = k_{m_b}/3$, $m_b(y) = k_{m_b}/2$, $m_b(z) = k_{m_b}/6$ in order to maintain the same relative belief of singletons

$$\tilde{b}(x) = 1/3, \quad \tilde{b}(y) = 1/2, \quad \tilde{b}(z) = 1/6$$

in all three cases. Figure 3-top-left plots the L_2 distances in the probability simplex $\mathcal{P} = Cl(b_x, b_y, b_z)$

$$d(p,p') \doteq \sqrt{\sum_{w\in\Theta} |p(w) - p'(w)|^2}$$

of $BetP[b]$ and $\tilde{pl}_b$ from $\tilde{b}$ as a function of k_{m_b}, in both cases a) and b). As stated by Theorem 2, for quasi-Bayesian b.f.s ($k_{m_b} \to 1$) all approximations are the same. It is interesting to notice, however, that for the pignistic function the rate of convergence to $\tilde{b}$ is the same no matter how the mass is assigned to higher-size events, and is *constant*.
For $\tilde{pl}_b$, instead, the rate of convergence differs in the two cases and is actually

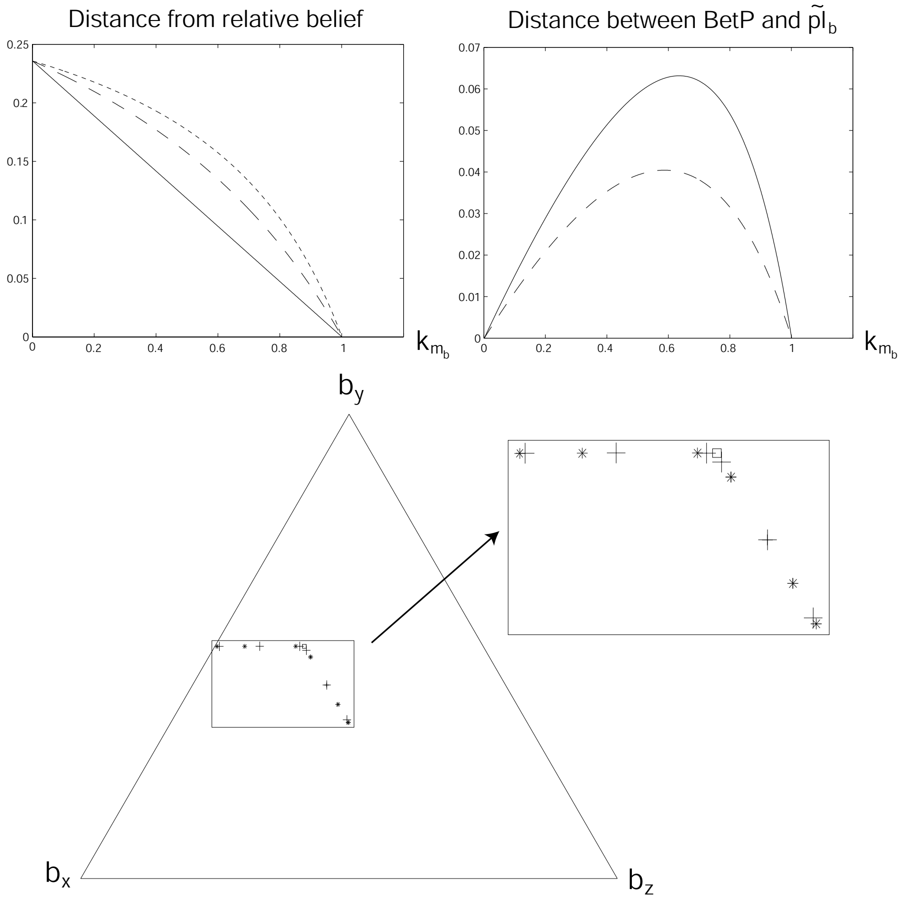

Fig. 3. Convergence of pignistic function and relative plausibility to the relative belief in the ternary frame $\Theta = \{x, y, z\}$. Top left: distance from $\tilde{b}$ of $BetP[b]$ (solid line) and $\tilde{pl}_b$ (dotted line: case a; dashed line: case b) as a function of k_{m_b}. Top right: Corresponding distance between $BetP$ and $\tilde{pl}_b$ (solid line: case a; dashed line: case b). Bottom: Sample locations in the probability simplex of $\tilde{b}$ (square), BetP[b] (stars) and $\tilde{pl}_b$ (crosses) for $k_{m_b} = 0.95$, $k_{m_b} = 0.5$, $k_{m_b} = 0.05$ in both case a) (towards the side b_x, b_y of the simplex) and b) (towards the barycenter of the simplex).

slower for *discounted* belief functions, i.e. b.f.s which assign all the mass of non-singletons to the whole frame Θ (case b), a rather counterintuitive result.

Figure 3-top-right plots by comparison the distance *between* $BetP[b]$ and $\tilde{pl}_b$ as a function of k_{m_b}, in the two cases (again: a - solid line, b - dashed line). The two Bayesian approximations turn out to be close for low values of k_{m_b} too (almost singular b.f.s) and their distance reaches a peak for intermediate values of the total mass of singletons. Such values are though different for the

two functions, and the divergence is reduced in the case of asymmetric mass assignment ($m_b(\{x,y\}) = 1 - k_{m_b}$).

Finally, Figure 3-bottom shows the location of all considered Bayesian approximations on the probability simplex in both cases (a and b) for the three sample values $k_{m_b} = 0.95$, $k_{m_b} = 0.5$, $k_{m_b} = 0.05$ of the total mass of singletons.

6 Conclusions

In this paper we discussed interpretations and applicability of the relative belief of singletons as a novel Bayesian approximation of a belief function. It has recently been proven that relative belief and plausibility of singletons form a distinct family of Bayesian approximations related to Dempster's rule, as they both commute with $\oplus$, and meet dual representation and idempotence properties [4]. Here we focused in particular on the semantics of rel.bel. On one side we stressed the analogy between the mechanisms generating relative belief and plausibility, pointing out that they correspond to antithetical estimates of the evidence supporting each singleton. We proved that $\tilde{b}$ is in fact equivalent to the relative plausibility of a plausibility (seen as a pseudo belief function), but that this symmetry is broken by the existence constraint acting on $\tilde{b}$. We argued though that situations in which the latter is not met are pathological, as all Bayesian approximations are forced to span a limited region of the probability simplex. Finally, we proved that, as all those approximations converge for quasi-Bayesian b.f.s, rel.bel. can be seen as a low-cost proxy to pignistic transformation and relative plausibility. The analysis of this convergence for different classes of b.f.s has provided us with some insight on the relation between the probabilities associated with a belief function.

Appendix

Proof of Theorem 1

By definition of b.pl.a. μ we have that

$$\sum_{A \supseteq \{x\}} \mu_b(A) = \sum_{A \supseteq \{x\}} (-1)^{|A|+1} \Big(\sum_{B \supseteq A} m_b(B) \Big) = \\ = - \sum_{B \supseteq \{x\}} m_b(B) \Big(\sum_{x \subseteq A \subseteq B} (-1)^{|A|} \Big)$$

where $\sum_{x \subseteq A \subseteq B} (-1)^{|A|} = 0$ if $B \neq \{x\}$, -1 if $B = \{x\}$ for Newton's binomial: $\sum_{k=0}^{n} 1^{n-k} (-1)^k = 0$.

Proof of Theorem 2

If $k_{m_b} \to 1$ then $\sum_{|A|>1} m_b(A) \to 0$ which implies

$$m_b(A) \to 0 \quad \forall A : |A| > 1$$

(as $m_b(A) \geq 0 \ \forall A$). But by definition of $BetP$, $\tilde{b}$ $\tilde{pl}$, we have that

$$BetP[b](x) \doteq m_b(x) + \sum_{A \supsetneq \{x\}} m_b(A) \to m_b(x),$$
$$\tilde{b}(x) \doteq \frac{m_b(x)}{k_{m_b}} \to m_b(x),$$
$$\tilde{pl}_b(x) \doteq \frac{pl_b(x)}{\sum_x pl_b(x)} = \frac{m_b(x) + \sum_{A \supsetneq \{x\}} m_b(A)}{\sum_x (m_b(x) + \sum_{A \supsetneq \{x\}} m_b(A))} \to \frac{m_b(x)}{k_{m_b}} = m_b(x).$$

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