The Total Belief Theorem

UAI2017 ID #56

Chunlai Zhou & Fabio Cuzzolin

Renmin University of China, Oxford Brookes University

czhou@ruc.edu.cn, fabio.cuzzolin@brookes.ac.uk

Abstract

In this paper, motivated by the treatment of conditional constraints in the data association problem, we state and prove the generalisation of the law of total probability to belief functions, as finite random sets. Our results apply to the case in which Dempster's conditioning is employed. We show that the solution to the resulting total belief problem is in general not unique, whereas it is unique when the a-priori belief function is Bayesian. Examples and case studies underpin the theoretical contributions. Finally, our results are compared to previous related work on the generalisation of Jeffrey's rule by Spies and Smets.

Belief Functions

Let Ω be a frame of discernment. A *mass assignment* (or *mass function*) over Ω is a mapping $m : \mathcal{A} \to [0,1]$ satisfying $\sum_{A\in\mathcal{A}} m(A) = 1$. A *belief function* is a function $bel : \mathcal{A} \to [0,1]$ satisfying the conditions: $bel(\emptyset) = 0$, $bel(\Omega) = 1$ and $bel(\bigcup_{i=1}^n A_i) \geq \sum_{\emptyset\neq I\subseteq\{1,\cdots,n\}}(-1)^{|I|+1} bel(\cap_{i\in I} A_i)$ where $A_i \subseteq \Omega$ for all $i \in \{1,\cdots,n\}$. It can be obtained from m as follows: $bel(A) = \sum_{B\subseteq A} m(B)$ for all $A \in \mathcal{A}$ [2].

- The belief function bel is called *Bayesian* if $m(A) = 0$ for all non-singletons A. It is called *categorical* if it has only one focal set. And it is called *vacuous* if Ω is the only focal element. The vacuous belief represents the state of total ignorance.
- If m_1 and m_2 are two mass functions on Ω induced by two *independent* evidential sources, the combined mass function is calculated according to *Dempster's rule of combination*: for any $C \subseteq \Omega$,

$$(m_1 \oplus m_2)(C) = \frac{\sum_{A\cap B=C} m_1(A)m_2(B)}{\sum_{A\cap B\neq\emptyset} m_1(A)m_2(B)}$$

  When an event E is observed, then the conditional mass function of m is obtained according to *Dempster conditioning*: for any $C \subseteq \Omega$, $m(C \mid E) = \frac{\sum_{B\cap E=C} m(B)}{pl(E)}$.
- Suppose that Θ is a *finer* frame than Ω. This means that the elements $\omega_1, \cdots, \omega_{|\Omega|}$ of Ω correspond to a partition $\Pi_1, \cdots, \Pi_{|\Omega|}$ of Θ: a subset $\{\omega_{i_1}, \cdots \omega_{i_k}\}$ of Ω has the same meaning as the subset $\Pi_{i_1} \cup \cdots \cup \Pi_{i_k}$ of Θ. This identification can represented by a mapping $\rho : 2^\Omega \to 2^\Theta$ such that $\rho(\{\omega_i\}) = \Pi_i (1 \leq i \leq |\Omega|)$ and $\rho(\{\omega_{i_1}, \cdots \omega_{i_k}\}) = \cup_{j=1}^k \rho(\omega_{i_j}) = \cup_{j=1}^k \Pi_{i_j}$.

Motivating Problem

In data association we are given a sequence of images $\{I(k), k\}$, each containing a number of feature points $\{t_i(k)\}$ at time k which are projections of real world targets $\{T_1, ..., T_M\}$. We seek the correspondences $t_i(k) \longleftrightarrow t_j(k+1)$ between feature points in consecutive images which are projections of the same target [1].

If we assume that targets represent fixed positions on an articulated body connected by a rigid link, we can address the association task in critical situations in which several targets coalesce (*model-based* data association) via a set of logical constraints on the admissible positions of the targets. We can identify, among others:

(i) a 'prediction' constraint which encodes the likelihood of a measurement in the current image being associated with a measurement of the past image (e.g. produced by a Kalman filter in joint probabilistic data association);

(ii) a *rigid motion* constraint, acting on pairs of targets T_{j_1}, T_{j_2} connected by a rigid link in the model:

$$\|t_i(k) - t_j(k)\| \cong \|t_{i'}(k-1) - t_{j'}(k-1)\|,$$

assuming that $t_i(k), t_{i'}(k-1)$ are both projections of T_{j_1}, and $t_j(k), t_{j'}(k-1)$ are images of T_{j_2}. All such constraints can be expressed as belief functions over a suitable frame of discernment.

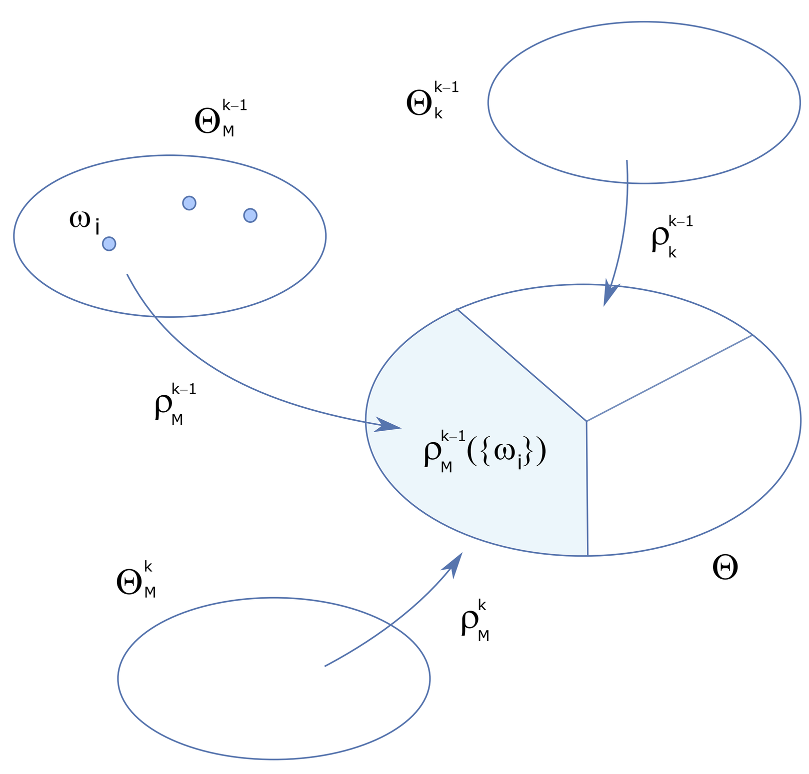

Figure 1: The family of past and present association frames. All the constraints of the model-based association problem are combined over the common refinement Θ and then marginalized onto the current association frame Θ_M^k to yield a belief estimate of the current feature-to-model association.

Formulation

Abstracting from the data association problem, we can then state the conditions an overall, total belief function b must obey, given a set of conditional belief functions $b_i : 2^{\Pi_i} \to [0,1]$ over the elements Π_i of the partition $\Pi = \{\Pi_1, ..., \Pi_{|\Omega|}\}$ of a frame Θ induced by a coarsening Ω.

1. *A-priori constraint*: the marginal on the coarsening Ω of the frame Θ of the candidate total belief function b must coincide with a given *a-priori* b.f. $b_0 : 2^\Omega \to [0,1]$.

In the data association problem the *a-priori* constraint is represented by the BF encoding the estimate of the past feature-to-model association $M \leftrightarrow m(k-1)$, defined over Θ_k^{k-1} (Figure 1). It ensures that the belief total function is compatible with the last available estimate.

2. *Conditional constraint*: the belief function $b(\cdot|\Pi_i)$ obtained by (Dempster's) conditioning the total belief function b with respect to each element Π_i of the partition Π must coincide with the corresponding given conditional belief function b_i:

$$b(\cdot|\Pi_i) = b \oplus b_{\Pi_i} = b_i \quad \forall i = 1, ..., N,$$

where $m_{\Pi_i} : 2^\Theta \to [0,1]$ is such that:

$$m_{\Pi_i}(A) = \begin{cases} 1 & A = \Pi_i \\ 0 & A \subseteq \Theta, A \neq \Pi_i. \end{cases}$$

Main Theorem

Theorem 1. *Suppose Θ and Ω are two frames of discernment, and $\rho : 2^\Omega \to 2^\Theta$ a given refining between them. Let b_0 be a belief function defined over $\Omega = \{\omega_1, ..., \omega_{|\Omega|}\}$. Suppose there exists a collection of belief functions $b_i : 2^{\Pi_i} \to [0,1]$, where $\Pi = \{\Pi_1, ..., \Pi_{|\Omega|}\}$, $\Pi_i = \rho(\{\omega_i\})$, is the partition of Θ induced by its coarsening Ω. Then, there exists a total belief function $b : 2^\Theta \to [0,1]$ such that:*

- *(P1) $b \oplus b_{\Pi_i} = b_i$ $\forall i = 1, ..., |\Omega|$, where b_{Π_i} is the categorical belief function with mass m_{Π_i} (1);*
- *(P2) b_0 is the marginal of b on Ω, $b_0 = b \upharpoonright_\Omega$.*

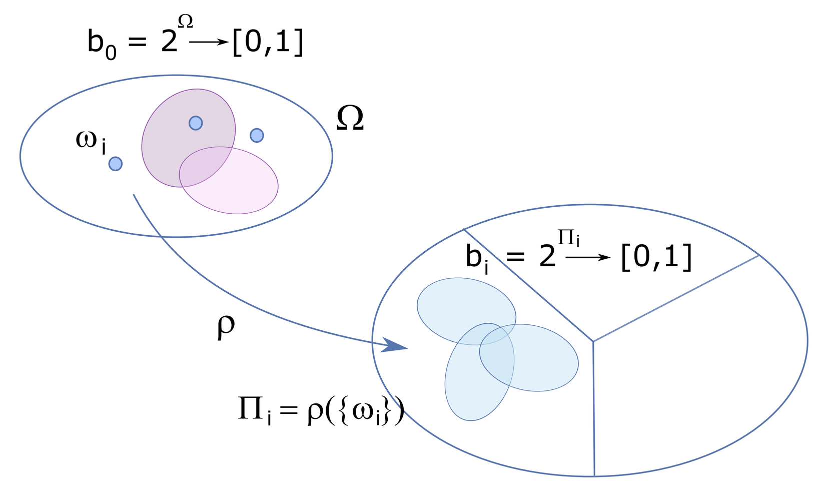

Figure 2: Total Belief Theorem

- The total belief function b obtained in Theorem 1 is *not unique*.
- However, for the belief function b_0 over Ω and conditional belief functions b_i over Π_i in Theorem 1, if b_0 is Bayesian (a probability function) such that $b_0(\omega_i) > 0$ for all $1 \leq i \leq |\Omega|$, then there is a unique total belief function $b : 2^\Theta \to [0,1]$ such that:
  1. $b \oplus b_{\Pi_i} = b_i$ for all $i = 1, \cdots, |\Omega|$ where b_{Π_i} is the categorical belief function with $m_{\Pi_i}(A) = 1$ if $A = \Pi_i$, and is 0, o.w.;
  2. b_0 is the marginal of b on Ω, i.e., $b_0 = b \upharpoonright_\Omega$.

  Moreover, the total mass function m of b is:

$$m(e) = \begin{cases} m_i(e)b_0(\omega_i) & \text{if } e \in \mathcal{E}_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Example

Suppose that the considered coarsening $\Omega := \{\omega_1, \omega_2, \omega_3\}$ induces a partion Π of Θ: $\{\Pi_1, \Pi_2, \Pi_3\}$. Also suppose that the considered conditional belief function b_1 defined on Π_1 has two focal elements e_1^1 and e_1^2; the conditional belief function b_2 defined on Π_2 has a single focal element e_2^1; b_3 defined on Π_3 has two focal elements e_3^1 and e_3^2 (See Figure 3).

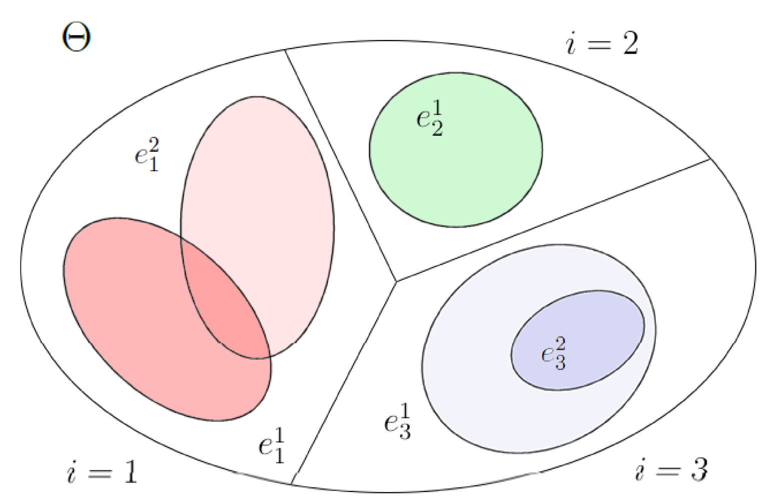

Figure 3: Conditional Belief Functions

- If we assume that a prior b_0 on Ω has each subset of Ω as a focal element, then the set $\mathcal{E}$ of the focal elements of b is $\{e = \bigcup_{1\leq i\leq I} e_i : 1 \leq I \leq 3, e_i \in \mathcal{E}_i\}$.
- It is very *straightforward* to compute the corresponding total mass function m. For example, for the two focal elements $e_1^1 \cup e_3^2$ and $e_1^1 \cup e_2^1 \cup e_3^2$ we have:

$$m(e_1^1 \cup e_3^2) = m_0(\{w_1, w_3\})m_1(e_1^1)m_3(e_3^2)$$
$$m(e_1^1 \cup e_2^1 \cup e_3^2) = m_0(\Omega)m_1(e_1^1)m_2(e_2^1)m_3(e_3^2).$$

Conclusions

- In this paper we stated and proved the generalisation of the law of total probability to belief measures, for the case in which Dempster's conditioning is employed. We showed that the solution is not unique, whereas it is unique when the a-priori belief function is Bayesian.
- These results can be further extended in a number of ways, especially the relationship between marginal extension and the law of total belief needs therefore to be understood.
- Finally, fascinating relationships exist between the total belief problem and transversal matroids, on one hand, and the theory of positive linear systems, on the other, as hinted at in this paper, which will be investigated in the near future.

References

[1] F. Cuzzolin, *Visions of a generalized probability theory*, Lambert Academic Publishing, 2014.

[2] G. Shafer, *A mathematical theory of evidence*, Princeton University Press, 1976.