Families of compatible frames of discernment as semimodular lattices

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Shafer's mathematical theory of evidence is one of the most remarkable attempts to introduce a complete generalized probability theory.
One of its features is the central role of the notion of different level of knowledge of a given phenomenon, embodied into the idea of *compatible frames of discernment*.[2]
In this work we are going to analyze the algebraic basement[1] of the concept of family of frames

$$\mathcal{F} = \{\mathcal{S}, \mathcal{R}\}.$$

The double structure composed by frames $\mathcal{S} = \{\Theta\}$ and refinings $\mathcal{R} = \{\omega : 2^{\Theta}_1 \rightarrow 2^{\Theta}_2, \; \Theta_i \in \mathcal{S}\}$ is discussed and the correspondence between them analyzed.
We will distinguish among *complete*, *finite generated* and *general* families of frames. The *monoidal* and *modular* structure of these collections is proved by discussing the properties of the internal operation of minimal refinement $\otimes$. By recalling the equivalence lattice of all the partitions of a set, we will introduce the *lattice*[3] structure by defining a *dual* operation called *maximal coarsening* $\oplus$ and proving its properties.
New axioms are introduced in order to reflect the principle of duality and give a constructive form to the theory. The equivalence among these different versions is easily proved. In addiction, the assumption of *finite knowledge* is shown to be the counterpart of finite generated families.
The analogy between the projective space of the linear subspaces of a finite dimensional vector space V and the notion of family of frames is discussed, with a particular attention to the idea of independence. Starting from the analysis of the *linear independence* relation among atoms of a semimodular lattice[3] we will introduce a relation

$$\{p_1, ..., p_n\} \, \mathcal{LI} \Leftrightarrow h(p_1 \vee \cdots \vee p_n) = h(p_1) + ... + h(p_n), \quad p_1, ..., p_n \in L$$

(where $h(x)$ is the rank of $x \in L$) that generalizes to arbitrary elements of a Birkhoff lattice bounded below L. The equivalence between *internal* independence of a collection of frames $\Theta_1, ..., \Theta_n$ as Boolean subalgebras of their minimal refinement $\Theta_1 \otimes \cdots \otimes \Theta_n$ and their *external* independence as elements of a Birkhoff lattice bounded below is proved.

Keywords: *Theory of evidence, compatible frames, semimodular lattices, independence.*

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