PRICAI-08

Alternative formulations of the theory of evidence

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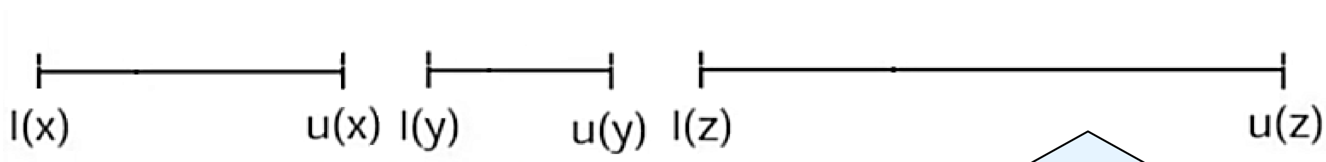
Outline

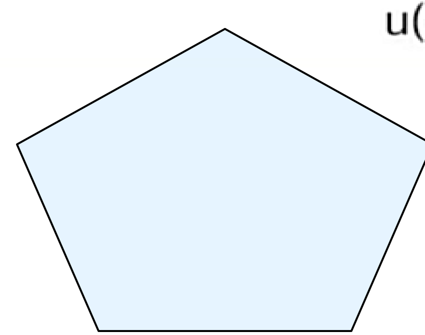
- Uncertainty measures
- Belief functions as sum functions
- Plausibilities and commonalities
- Geometric interpretation
- Plausibilities and commonalities as sum functions

Uncertainty measures

- classical probability: **probability distribution**
- realistic assumption: evidence is **not sufficient** to determine this probability
- **constraint** on this unknown, true probability
- different constrainst <-> different measures



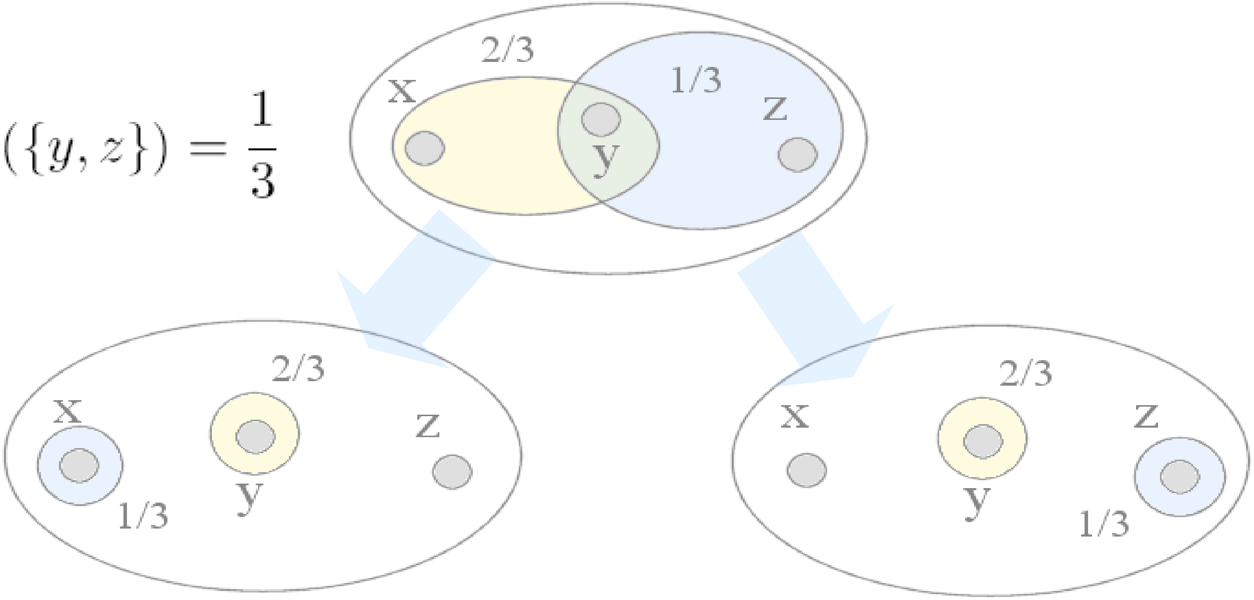
- interval probabilities, credal sets



Belief functions as constraints

- **belief function** → special case of constraint
- interpretation: mass m(A) can "float" inside A

$$m_b(\{x,y\}) = \frac{2}{3}, \qquad m_b(\{y,z\}) = \frac{1}{3}$$



- corresponds to a set of probabilities **consistent** with the belief function

Belief functions as sum functions

- given the **basic belief assignment** m ...
- the belief value of an event *A* is

$$b(A) = \sum_{B \subset A} m(B)$$

- *b* has the form of **sum function** (~integral)
- *m* is the **Moebius inverse** of *b* (~derivative)
- *m* is normalized and non-negative

Three equivalent formulations

- **belief function b(A)**
  - is the lower bound to the probability of A for a probability consistent with b

$$b(A) = \sum_{B \subseteq A} m_b(B)$$

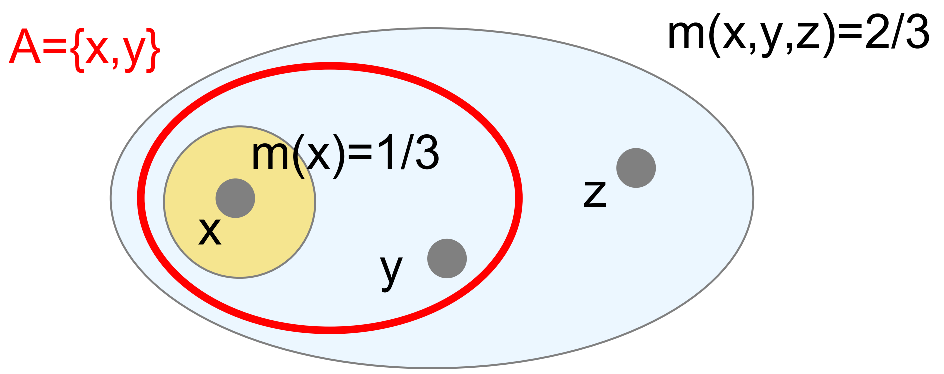
- **plausibility function pl(A)**
  - is the upper bound to the probability of A for a consistent probability

$$pl_b(A) \doteq \sum_{B \cap A \neq \emptyset} m_b(B)$$

- **commonality function Q(A)**
  - amount of evidence that equally supports all the elements of A

$$Q_b(A) \doteq \sum_{B \supseteq A} m_b(B)$$

Difference between belief, plausibility and commonality



- **b(x,y)**=$\sum_{A \subseteq \{x,y\}} m(A)$ = m(x)=**1/3** (sure support)
- **pl(x,y)**=$\sum_{A \cap \{x,y\} \neq \emptyset} m(A)$ = m(x)+m(x,y,z)=1/3+2/3=**1** (not surely against)
- **Q(x,y)**=$\sum_{A \supseteq \{x,y\}} m(A)$ = m(x,y,z)=**2/3**

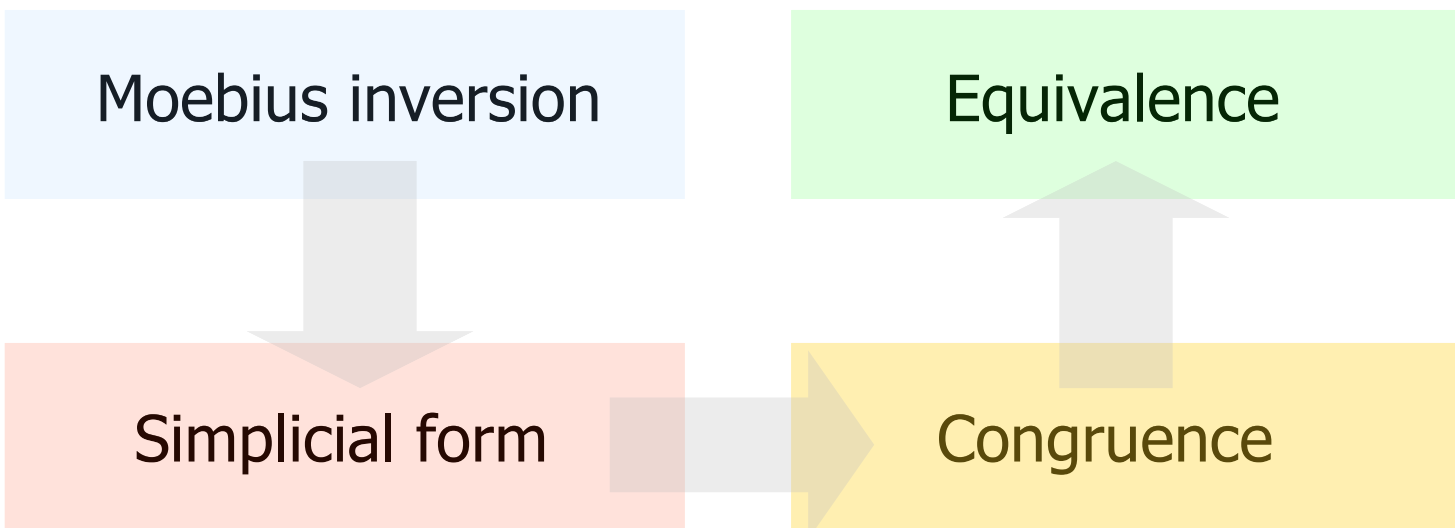
Moebius inversion

- belief function are sum functions $$b(A) = \sum_{B \subseteq A} m_b(B)$$
- analogous of integral in calculus
- **derivative = Moebius inversion**

belief function	$m_b(A) = \sum_{B \subseteq A} (-1)^{\|A-B\|} b(B)$	b.b.a.
plausibility function		?
commonality function		?

Geometric solution

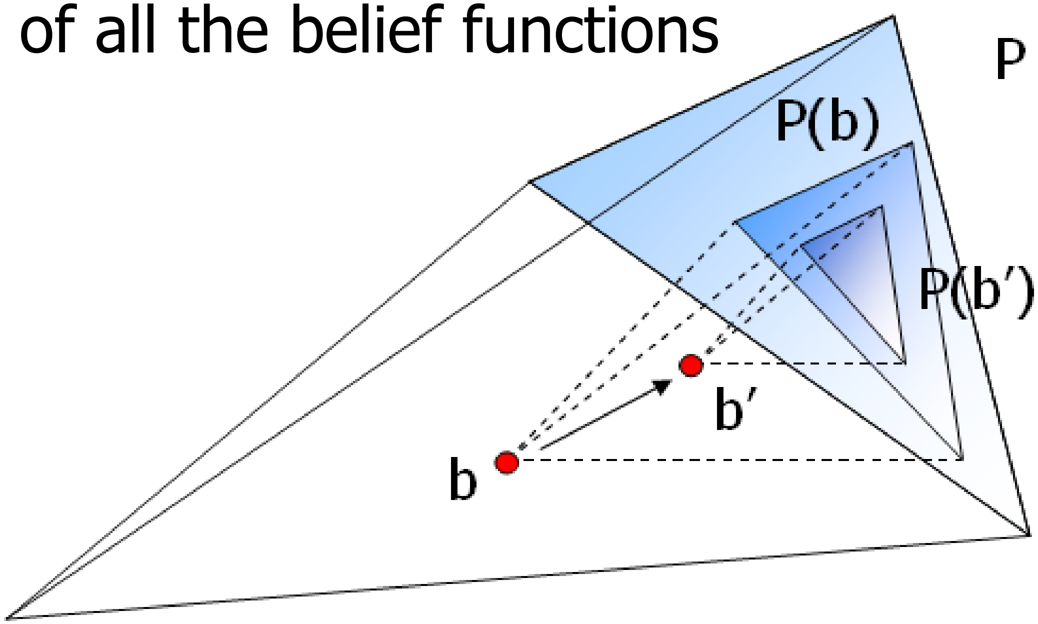
- how can you prove this?
- move the problem to a geometric setup



- Moebius inverses $\leftrightarrow$ simplicial coordinates
- equivalence of functions $\leftrightarrow$ congruence of simplices

A geometric approach to uncertainty

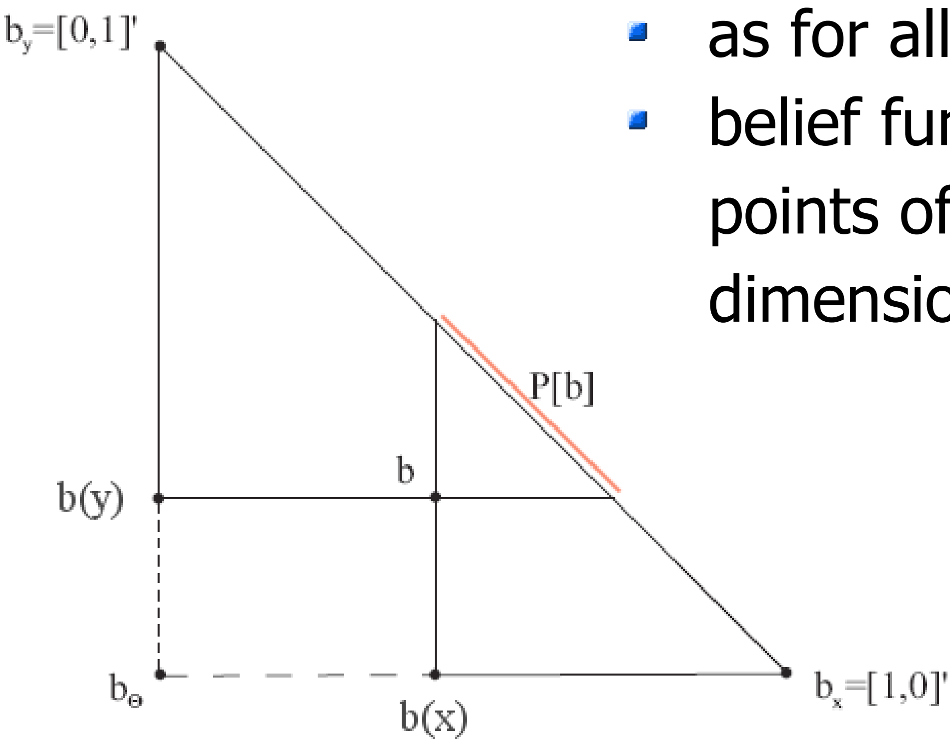
- belief space: the space of all the belief functions on a given frame

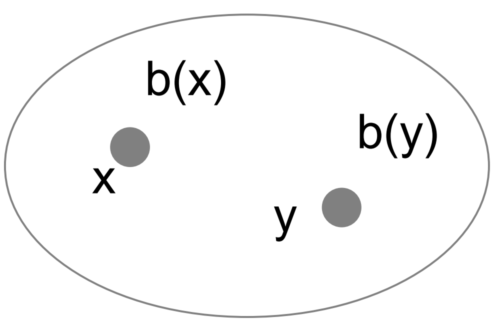


- it has the shape of a simplex

Belief functions as points

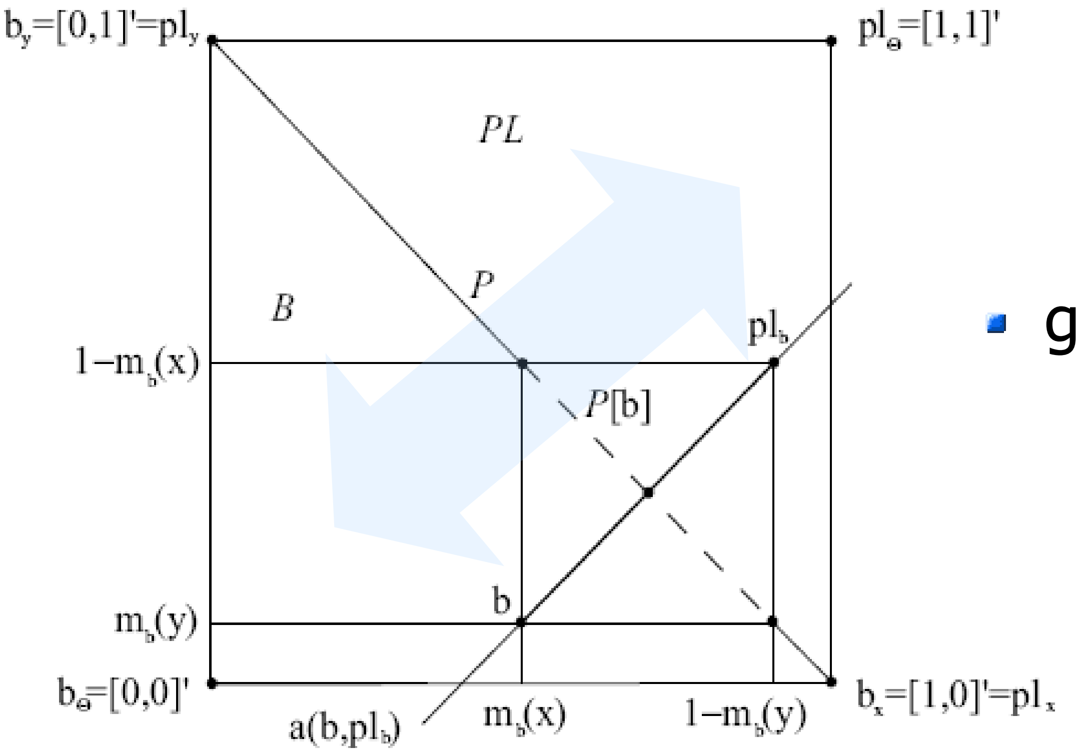
- if $n=|\Theta|=2$, a belief function b is specified by b(x) and b(y)
- as for all bfs, $b(\emptyset)=0$ and $b(\Theta)=1$
- belief functions can be seen as points of a Cartesian space of dimension 2^n-2





Congruent simplices

- plausibility functions pl(A) live in a simplex too

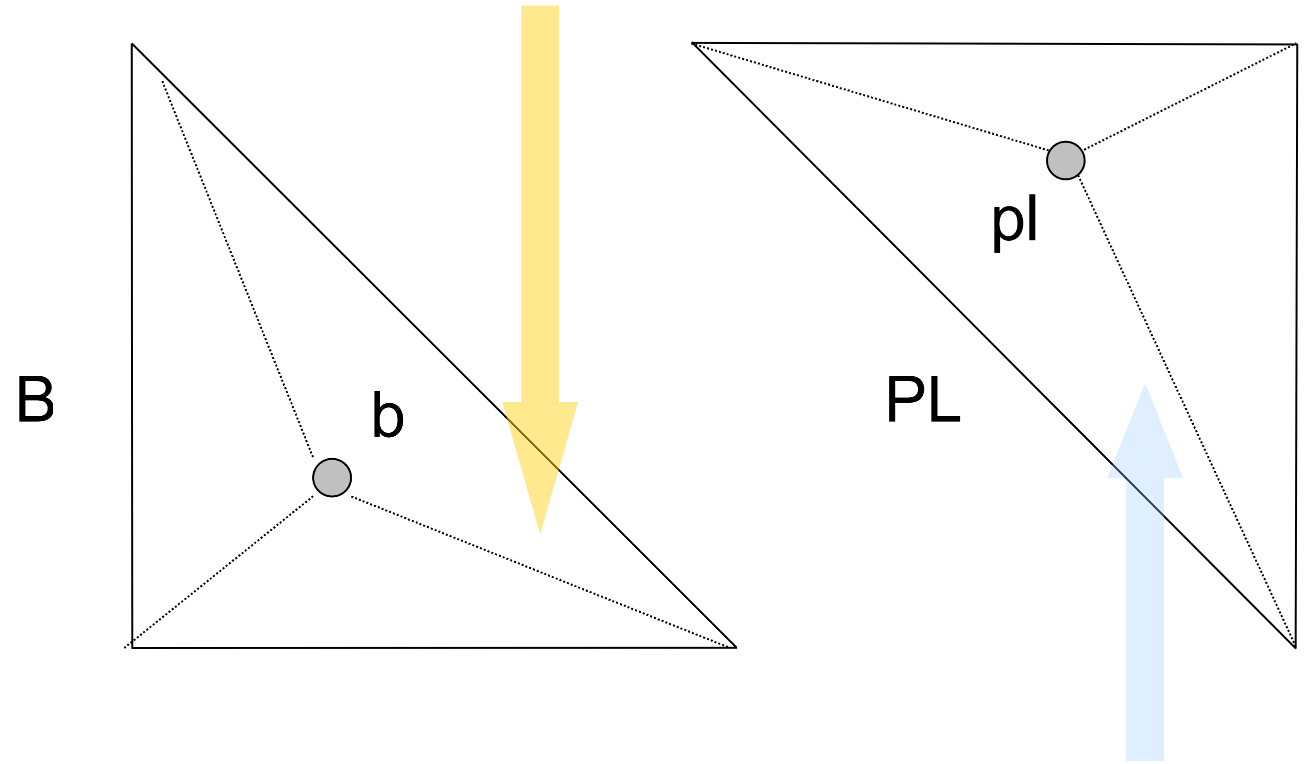


- geometrically, they form **congruent simplices**

- same is true for commonality functions Q(A)

Moebius inverse as simplicial coordinates

- coordinates of **b** in B given by the b.b.a. **m**



- coordinates of **pl** in PL given by its Moebius inverse μ

Equivalent theories

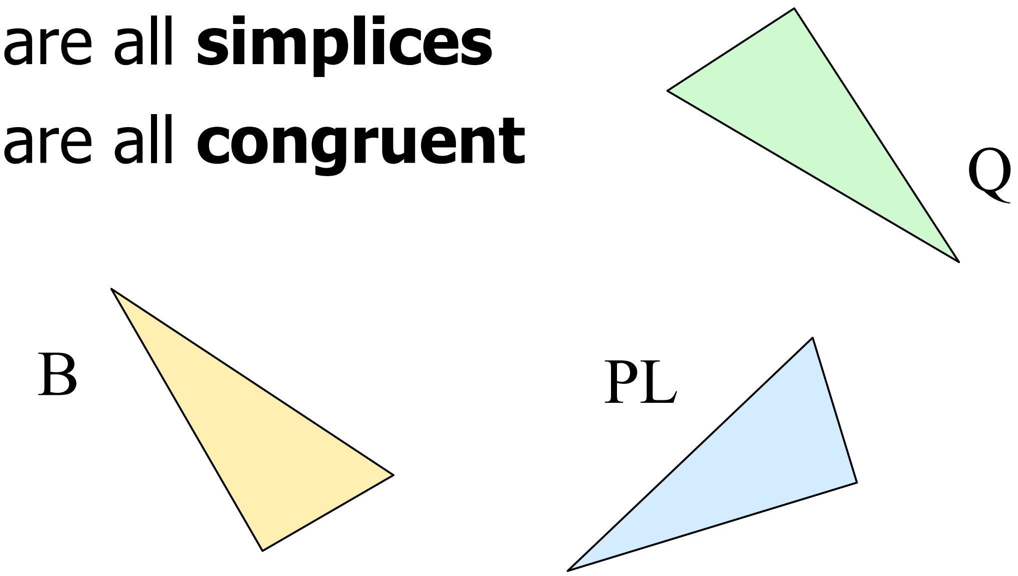
- they **all** have a Moebius inverse

belief function	$m_b(A) = \sum_{B \subseteq A} (-1)^{\lvert A-B \rvert} b(B)$	b.b.a.
plausibility function	$\mu_b(A) \doteq \sum_{B \subseteq A} (-1)^{\lvert A-B \rvert} pl_b(B)$	b.pl.a.
commonality function	$q_b(B) = \sum_{\emptyset \subseteq A \subseteq B} (-1)^{\lvert B \setminus A \rvert} Q_b(A)$	b.comm.a.

- **alternative formulations** of the theory can be given in terms of such assignments

Congruence and equivalence

- the spaces where belief, plausibility and commonality functions live ...
- ... are all **simplices**
- ... are all **congruent**

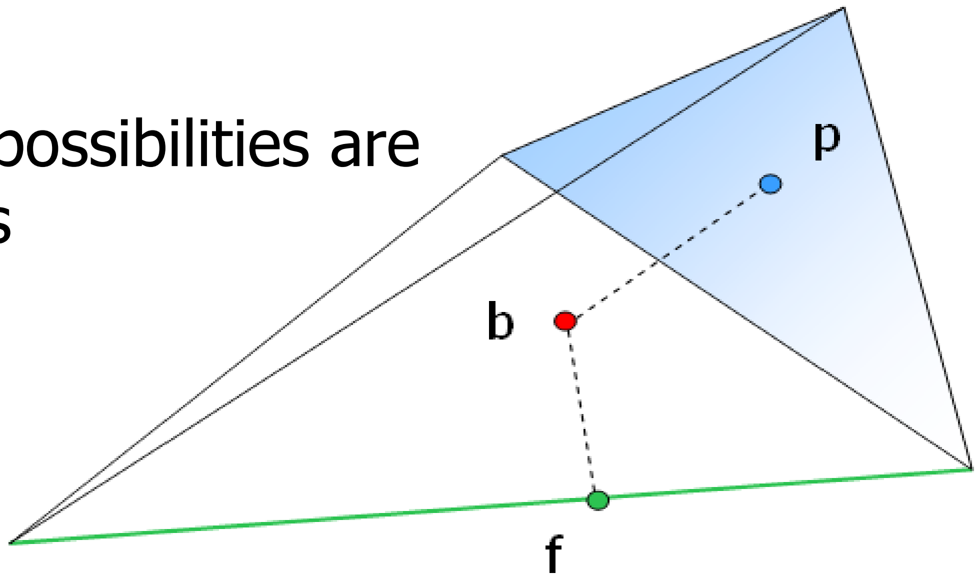


- geometrically, congruence is the counterpart of equivalence!

Approximation problem

- how to transform a measure of a certain family into a different uncertainty measure → can be done geometrically

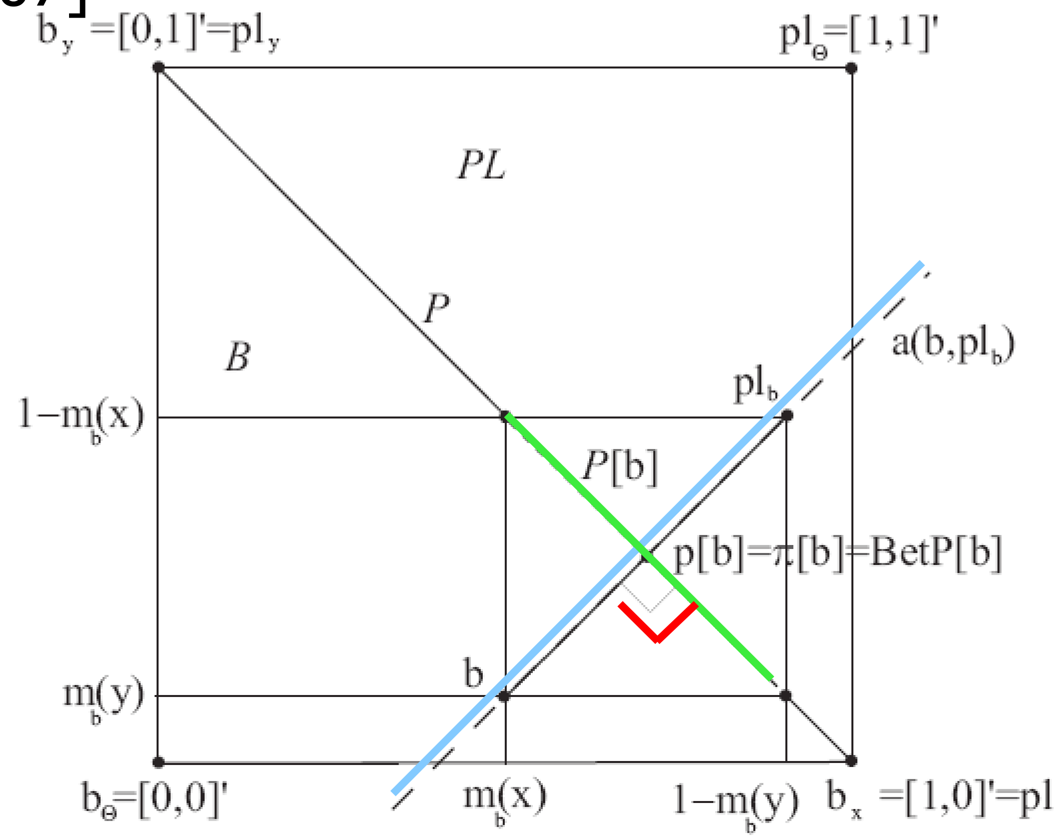
- probabilities, fuzzy sets, possibilities are all **special cases** of b.f.s



- IEEE Tr. SMC-B '07, IEEE Tr. Fuzzy Systems '07, AMAI '08, AI '08, IEEE Tr. Fuzzy Systems '08

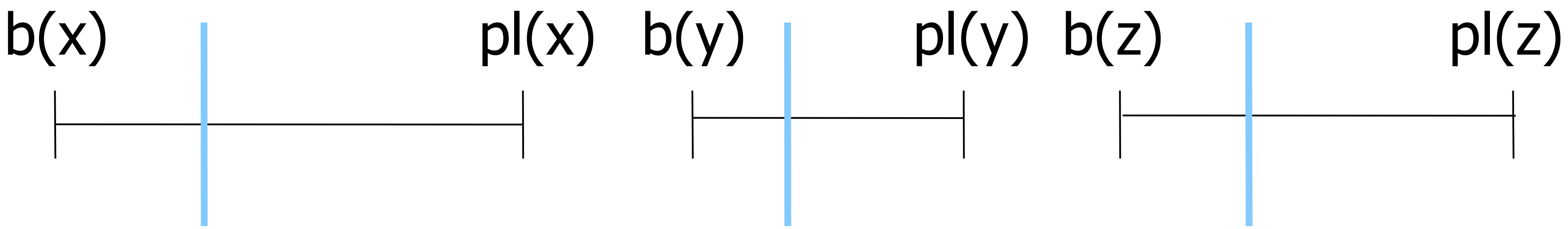
Geometric approximations

- the approximation problem can be posed in the geometric approach [IEEE SMC-B07]
- pignistic function BetP as barycenter of P[b]
- orthogonal projection π[b]
- intersection probability p[b]



Intersection probability

- it is derived from geometric arguments [IEEE SMC-B07]
- but is inherently associated with probability intervals



- it is the unique probability such that [AIJ08]

$$p(x) = b(x) + \alpha\,(pl(x) - b(x))$$

Conclusions

- belief functions are sum functions
- their Moebius inverse m gives their coordinates in a simplex
- plausibility and commonality functions carry the same information
- they also live in a simplex
- their coordinate there is their Moebius inverse
- they are also sum functions
- equivalence is congruence
- applications to approximation problem, decomposition