Alternative formulations of the theory of evidence

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Outline

- Uncertainty measures
- Belief functions as sum functions
- Plausibilities and commonalities
- Geometric interpretation
- Plausibilities and commonalities as sum functions
Uncertainty measures

- classical probability: probability distribution
- realistic assumption: evidence is not sufficient to determine this probability
- **constraint** on this unknown, true probability
- different constraints <-> different measures

- interval probabilities, credal sets
Belief functions as constraints

- **belief function** → special case of constraint
- interpretation: mass $m(A)$ can “float” inside $A$

$$m_b(\{x, y\}) = \frac{2}{3}, \quad m_b(\{y, z\}) = \frac{1}{3}$$

- corresponds to a set of probabilities **consistent** with the belief function
Belief functions as sum functions

• given the basic belief assignment \( m \) ...
• the belief value of an event \( A \) is

\[
b(A) = \sum_{B \subset A} m(B)
\]

• \( b \) has the form of sum function (~integral)
• \( m \) is the Moebius inverse of \( b \) (~derivative)
• \( m \) is normalized and non-negative
Three equivalent formulations

- **belief function** $b(A)$
  - is the lower bound to the probability of $A$ for a probability consistent with $b$

- **plausibility function** $pl(A)$
  - is the upper bound to the probability of $A$ for a consistent probability

- **commonality function** $Q(A)$
  - amount of evidence that equally supports all the elements of $A$
Difference between belief, plausibility and commonality

- $b(x,y) = \sum_{A \subseteq \{x,y\}} m(A) = m(x) = \frac{1}{3}$
  (sure support)

- $pl(x,y) = \sum_{A \cap \{x,y\} \neq \emptyset} m(A) = m(x) + m(x,y,z) = \frac{1}{3} + \frac{2}{3} = 1$
  (not surely against)

- $Q(x,y) = \sum_{A \supseteq \{x,y\}} m(A) = m(x,y,z) = \frac{2}{3}$
Moebius inversion

- belief function are sum functions
- analogous of integral in calculus
- **derivative = Moebius inversion**

\[
b(A) = \sum_{B \subseteq A} m_b(B)
\]

\[
m_b(A) = \sum_{B \subseteq A} (-1)^{|A-B|} b(B)
\]

b.b.a.

<table>
<thead>
<tr>
<th>belief function</th>
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<td>plausibility function</td>
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<td>commonality function</td>
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Geometric solution

- how can you prove this?
- move the problem to a geometric setup

- Moebius inverses $\leftrightarrow$ simplicial coordinates
- equivalence of functions $\leftrightarrow$ congruence of simplices
A geometric approach to uncertainty

- belief space: the space of all the belief functions on a given frame

- it has the shape of a simplex
Belief functions as points

- If $n=|\Theta|=2$, a belief function $b$ is specified by $b(x)$ and $b(y)$.
- As for all bfs, $b(\emptyset)=0$ and $b(\Theta)=1$.
- Belief functions can be seen as points of a Cartesian space of dimension $2^n-2$. 
Congruent simplices

- Plausibility functions $pl(A)$ live in a simplex too.
- Geometrically, they form congruent simplices.
- Same is true for commonality functions $Q(A)$. 
Moebius inverse as simplicial coordinates

- Coordinates of $b$ in $B$ given by the b.b.a. $m$

- Coordinates of $pl$ in $PL$ given by its Moebius inverse $\mu$
Equivalent theories

- they all have a Moebius inverse

**Belief function**

\[ m_b(A) = \sum_{B \subseteq A} (-1)^{|A-B|} b(B) \]

**Plausibility function**

\[ \mu_b(A) = \sum_{B \subseteq A} (-1)^{|A-B|} pl_b(B) \]

**Commonality function**

\[ q_b(B) = \sum_{\emptyset \subseteq C \subseteq B} (-1)^{|B\setminus A|} Q_b(A) \]

*alternative formulations* of the theory can be given in terms of such assignments
Congruence and equivalence

- the spaces where belief, plausibility and commonality functions live ...
- ... are all simplices
- ... are all congruent

- geometrically, congruence is the counterpart of equivalence!
how to transform a measure of a certain family into a different uncertainty measure → can be done geometrically

- probabilities, fuzzy sets, possibilities are all **special cases** of b.f.s

the approximation problem can be posed in the geometric approach [IEEE SMC-B07]

- pignistic function BetP as barycenter of $P[b]$  
- orthogonal projection $\pi[b]$  
- intersection probability $p[b]$
Intersection probability

- It is derived from geometric arguments [IEEE SMC-B07]
- But is inherently associated with probability intervals

\[ p(x) = b(x) + \alpha (pl(x) - b(x)) \]
Conclusions

- belief functions are sum functions
- their Moebius inverse $m$ gives their coordinates in a simplex
- plausibility and commonality functions carry the same information
- they also live in a simplex
- their coordinate there is their Moebius inverse
- they are also sum functions
- equivalence is congruence
- applications to approximation problem, decomposition