

PRICAI-08

Alternative formulations of the theory of evidence

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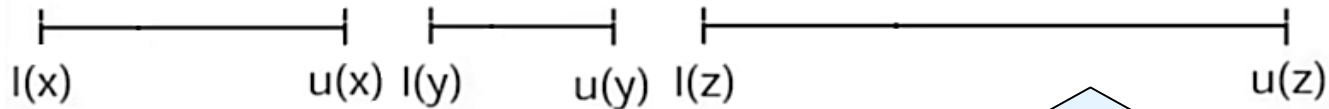
Outline

- Uncertainty measures
- Belief functions as sum functions
- Plausibilities and commonalities
- Geometric interpretation
- Plausibilities and commonalities as sum functions

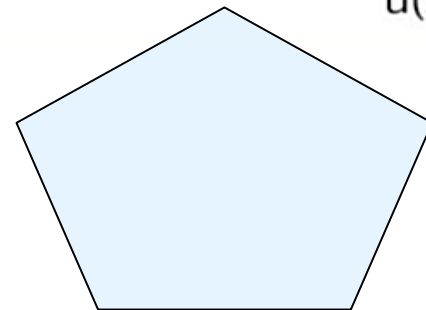


Uncertainty measures

- classical probability: **probability distribution**
- realistic assumption: evidence is **not sufficient** to determine this probability
- **constraint** on this unknown, true probability
- different constraint \leftrightarrow different measures



- interval probabilities, credal sets

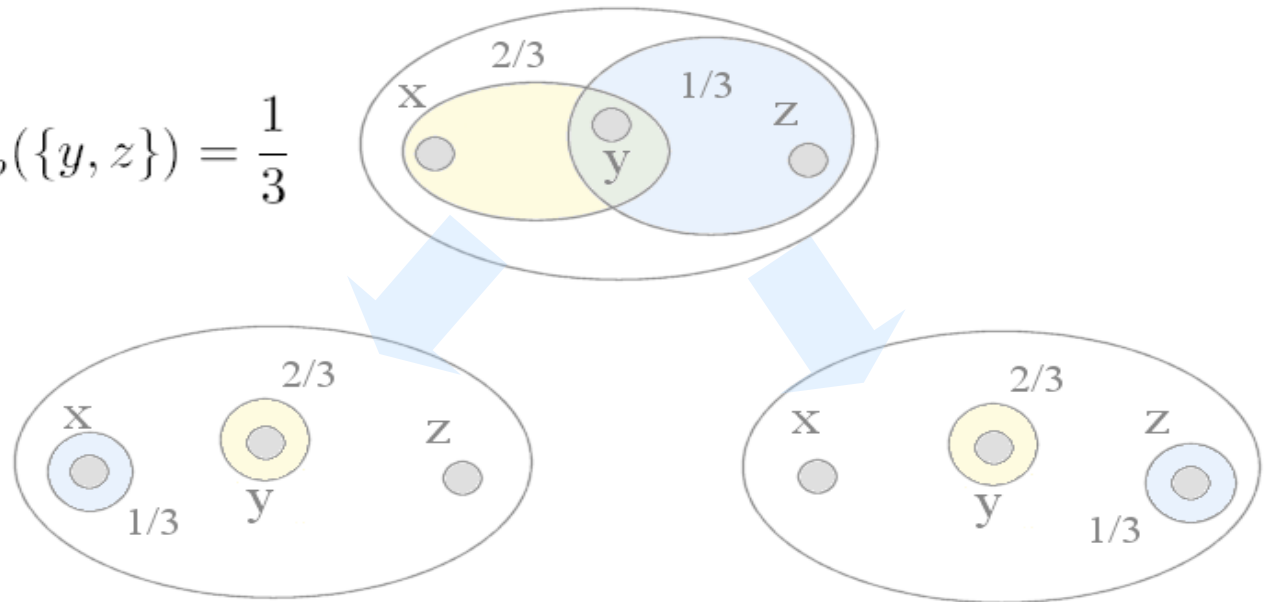


Belief functions as constraints

- **belief function** → special case of constraint
- interpretation: mass $m(A)$ can “float” inside A

$$m_b(\{x, y\}) = \frac{2}{3},$$

$$m_b(\{y, z\}) = \frac{1}{3}$$



- corresponds to a set of probabilities **consistent** with the belief function



Belief functions as sum functions

- given the **basic belief assignment** m ...
- the belief value of an event A is

$$b(A) = \sum_{B \subset A} m(B)$$

- b has the form of **sum function** (\sim integral)
- m is the **Moebius inverse** of b (\sim derivative)
- m is normalized and non-negative



Three equivalent formulations

- **belief function $b(A)$**

- is the **lower bound** to the probability of A for a probability consistent with b

$$b(A) = \sum_{B \subseteq A} m_b(B)$$

- **plausibility function $pl(A)$**

- is the **upper bound** to the probability of A for a consistent probability

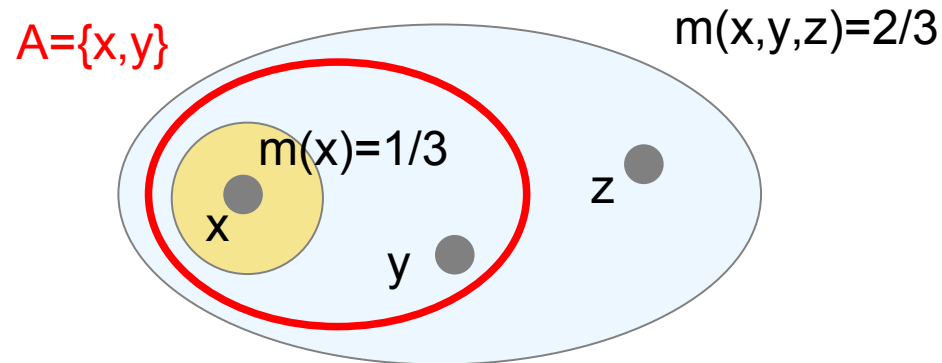
$$pl_b(A) \doteq \sum_{B \cap A \neq \emptyset} m_b(B)$$

- **commonality function $Q(A)$**

- amount of evidence that **equally supports** all the elements of A

$$Q_b(A) \doteq \sum_{B \supseteq A} m_b(B)$$

Difference between belief, plausibility and commonality



- $\mathbf{b(x,y)} = \sum_{A \subseteq \{x,y\}} m(A) = m(x) = \mathbf{1/3}$
(sure support)
- $\mathbf{pl(x,y)} = \sum_{A \cap \{x,y\} \neq \emptyset} m(A) = m(x) + m(x,y,z) = 1/3 + 2/3 = \mathbf{1}$
(not surely against)
- $\mathbf{Q(x,y)} = \sum_{A \supseteq \{x,y\}} m(A) = m(x,y,z) = \mathbf{2/3}$

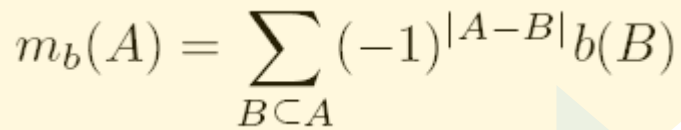


Moebius inversion

- belief function are sum functions
- analogous of integral in calculus
- **derivative = Moebius inversion**

$$b(A) = \sum_{B \subseteq A} m_b(B)$$

belief function


$$m_b(A) = \sum_{B \subseteq A} (-1)^{|A-B|} b(B)$$

b.b.a.

plausibility
function

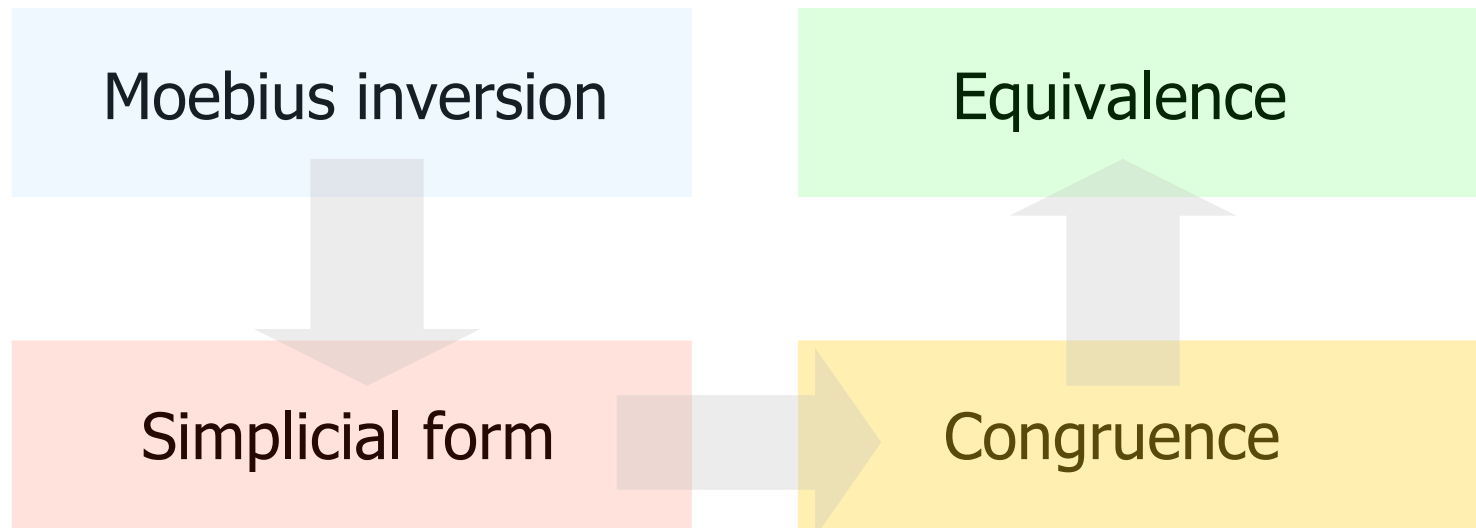
?

commonality
function

?

Geometric solution

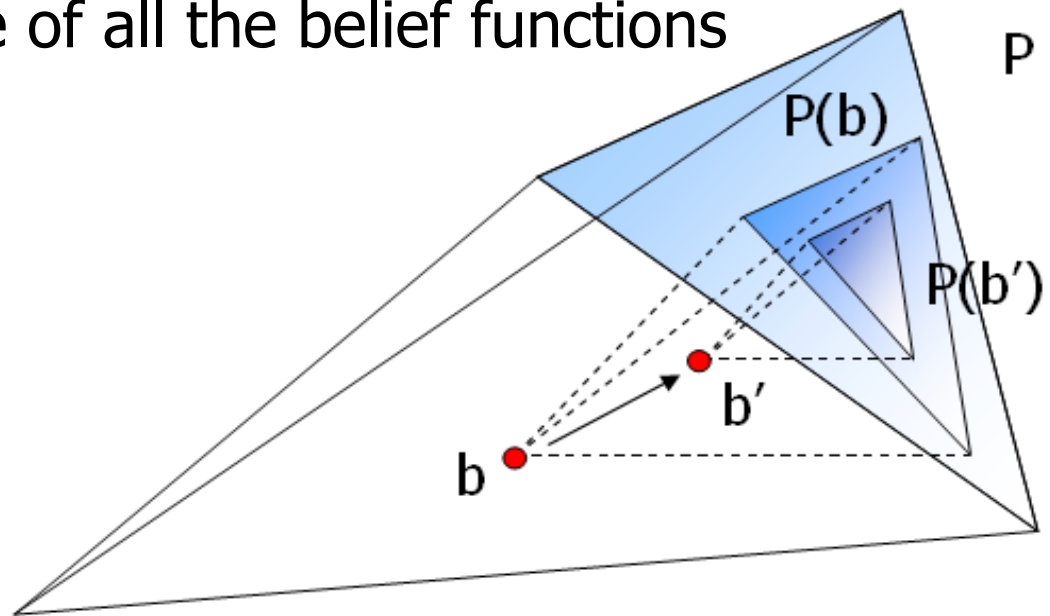
- how can you prove this?
- move the problem to a geometric setup



- Moebius inverses \leftrightarrow simplicial coordinates
- equivalence of functions \leftrightarrow congruence of simplices

A geometric approach to uncertainty

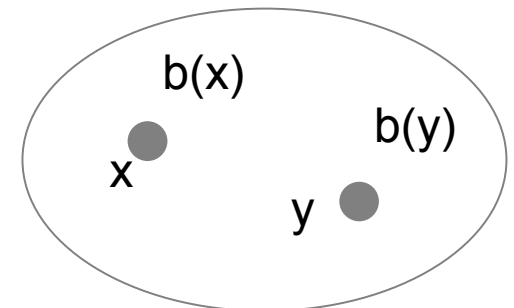
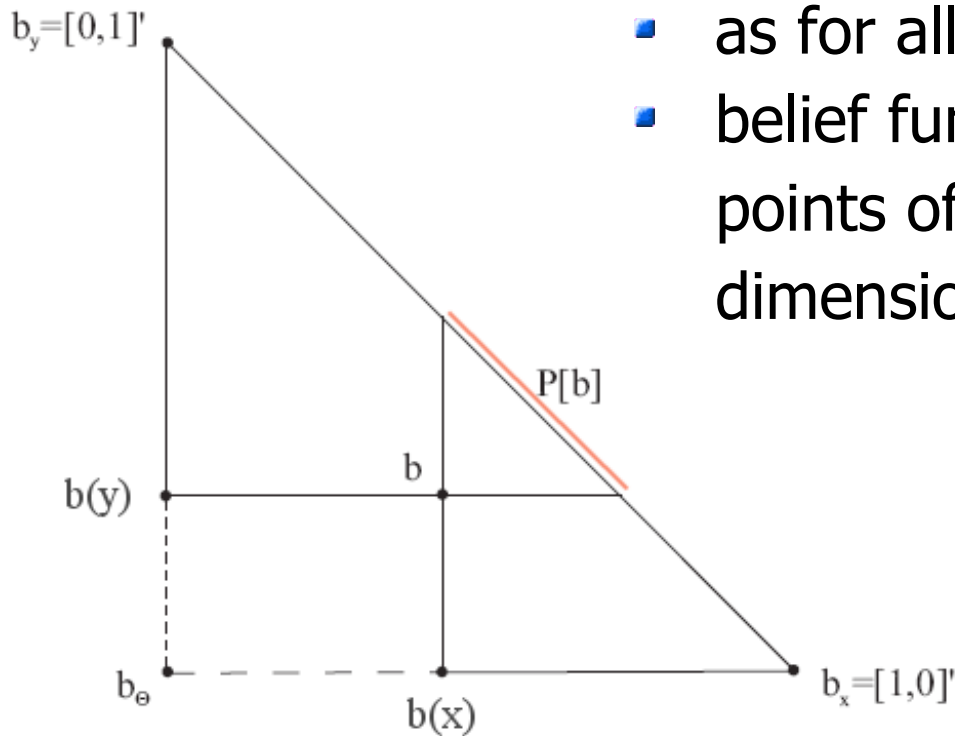
- belief space: the space of all the belief functions on a given frame



- it has the shape of a simplex

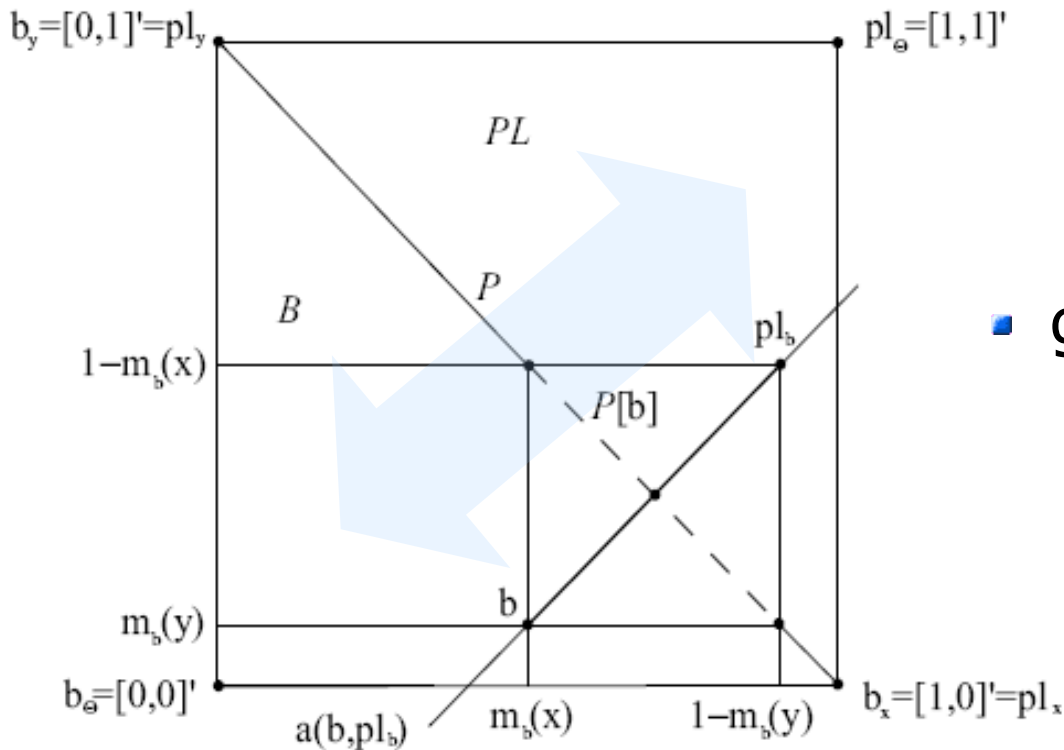
Belief functions as points

- if $n=|\Theta|=2$, a belief function b is specified by $b(x)$ and $b(y)$
- as for all bfs, $b(\emptyset)=0$ and $b(\Theta)=1$
- belief functions can be seen as points of a Cartesian space of dimension 2^n-2



Congruent simplices

- plausibility functions $pl(A)$ live in a simplex too

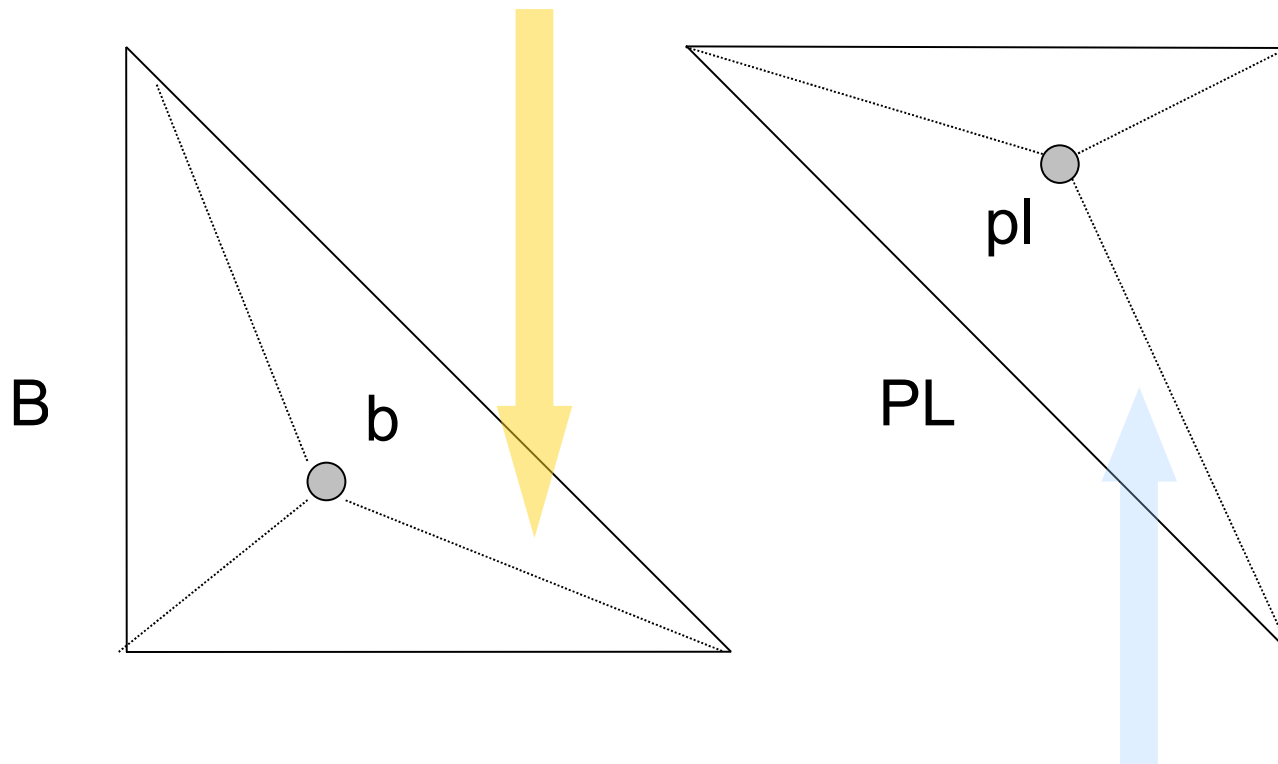


- geometrically, they form **congruent simplices**

- same is true for commonality functions $Q(A)$

Moebius inverse as simplicial coordinates

- coordinates of **b** in B given by the b.b.a. **m**



- coordinates of **pl** in PL given by its Moebius inverse μ



Equivalent theories

- they **all** have a Moebius inverse

belief function

$$m_b(A) = \sum_{B \subseteq A} (-1)^{|A-B|} b(B)$$

b.b.a.

plausibility
function

$$\mu_b(A) \doteq \sum_{B \subseteq A} (-1)^{|A-B|} pl_b(B)$$

b.pl.a.

commonality
function

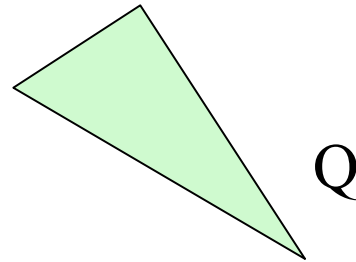
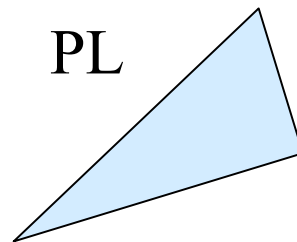
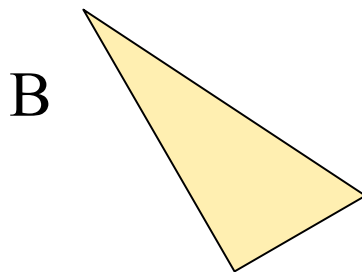
$$q_b(B) = \sum_{\emptyset \subseteq A \subseteq B} (-1)^{|B \setminus A|} Q_b(A)$$

b.comm.a.

- alternative formulations** of the theory can be given in terms of such assignments

Congruence and equivalence

- the spaces where belief, plausibility and commonality functions live ...
- ... are all **simplices**
- ... are all **congruent**

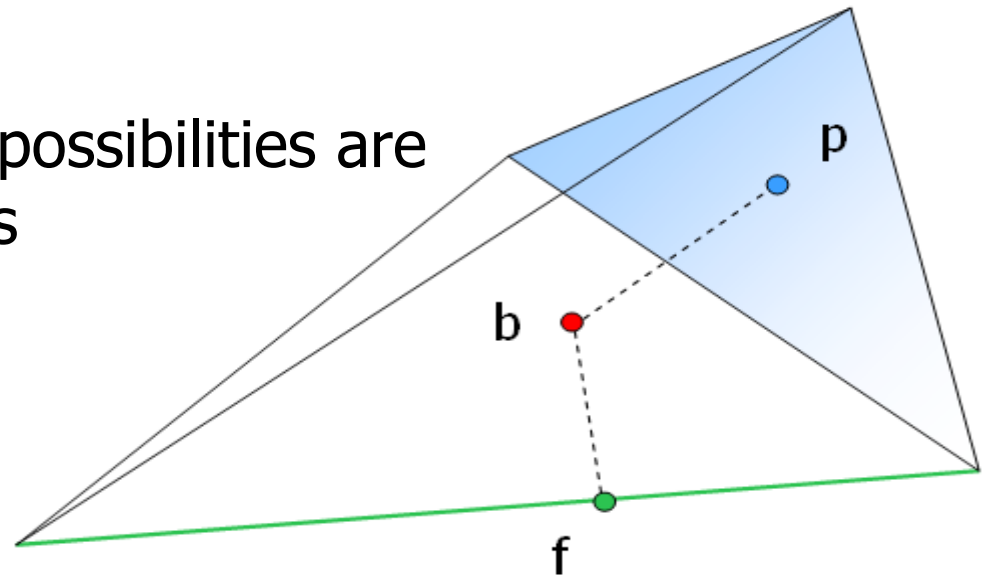


- geometrically, congruence is the counterpart of equivalence!

Approximation problem

- how to transform a measure of a certain family into a different uncertainty measure → can be done geometrically

- probabilities, fuzzy sets, possibilities are all **special cases** of b.f.s



- IEEE Tr. SMC-B '07, IEEE Tr. Fuzzy Systems '07, AMAI '08, AI '08, IEEE Tr. Fuzzy Systems '08

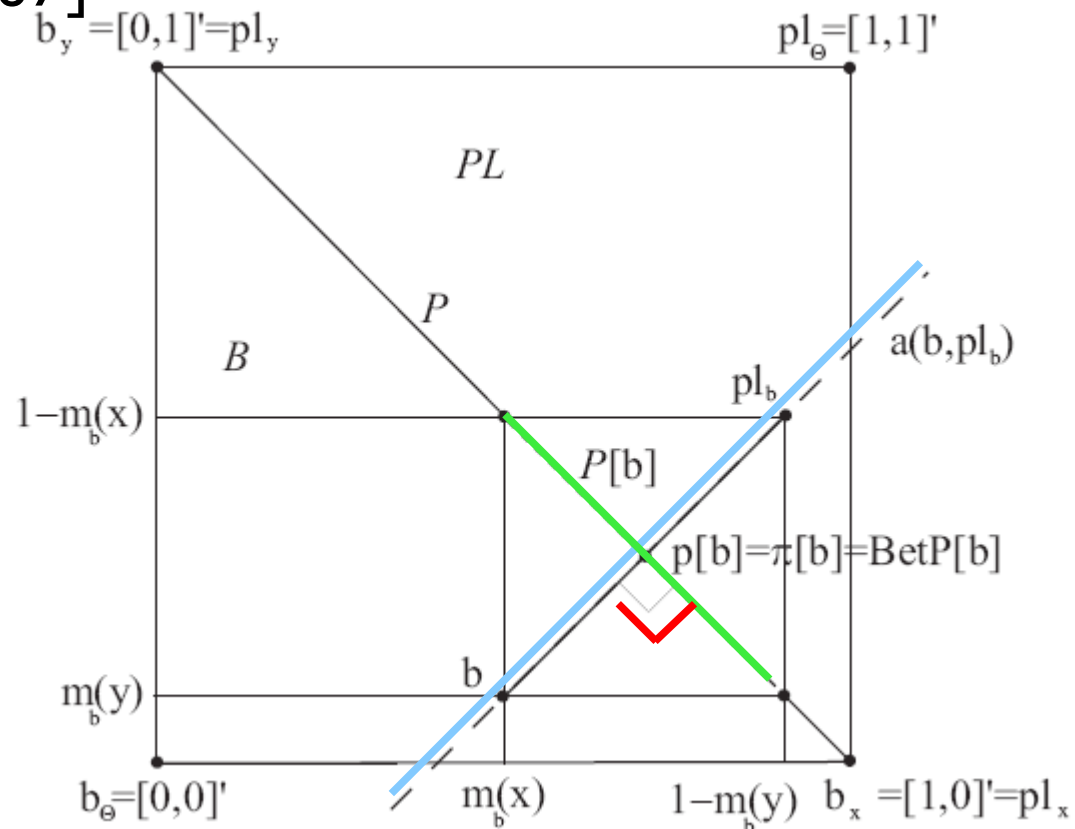
Geometric approximations

- the approximation problem can be posed in the geometric approach [IEEE SMC-B07]

- pignistic function BetP as barycenter of $P[b]$

- orthogonal projection $\pi[b]$

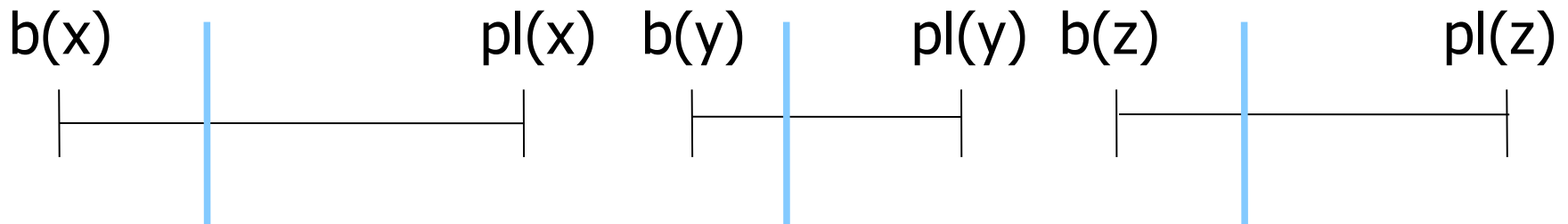
- intersection probability $p[b]$





Intersection probability

- it is derived from geometric arguments [IEEE SMC-B07]
- but is inherently associated with probability intervals



- it is the unique probability such that [AIJ08]

$$p(x) = b(x) + \alpha (pl(x) - b(x))$$



Conclusions

- belief functions are sum functions
- their Moebius inverse m gives their coordinates in a simplex
- plausibility and commonality functions carry the same information
- they also live in a simplex
- their coordinate there is their Moebius inverse
- they are also sum functions
- equivalence is congruence
- applications to approximation problem, decomposition