Dual properties of the relative belief of singletons

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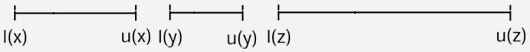
Outline

Uncertainty measures

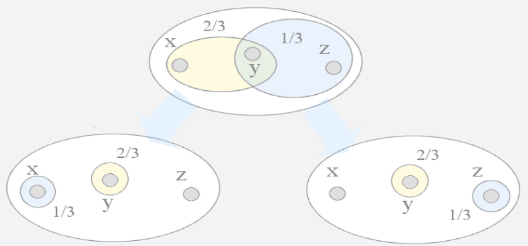
- classical probability: uncertainty is described by a **probability distribution**
- realistic assumption: evidence is not sufficient to determine this probability
- **constraint** on the unknown, true probability
- different constraint $\leftrightarrow$ different uncertainty measure



- simplest example: interval probability

Belief functions as constraints

- **belief function** $\rightarrow$ a special case of constraint on probability distributions



- corresponds to a **convex set of probability measures**
- completely described by $m(\{x, y\}) = 2/3$, $m(\{y, z\}) = 1/3$

Basic belief assignment

- **basic belief assignment $\leftrightarrow$ belief functions**
  - basic belief assignment $m : 2^\Theta \to [0,1]$ such that

$$m(\emptyset) = 0, \sum_{A \subseteq \Theta} m(A) = 1, m(A) \geq 0 \ \forall A \subseteq \Theta$$

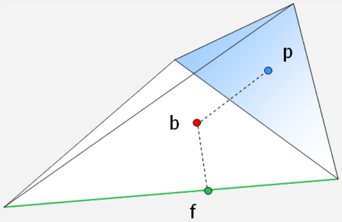
  - *belief function* $b : 2^\Theta \to [0,1]$,

$$b(A) = \sum_{B \subseteq A} m(B)$$

- probability functions are a **special case** of belief functions (Bayesian b.f.s): $m(A) = 0$, $|A| > 1$
- fuzzy sets are also a special case

Probabilistic transformation

- problem: **finding the probability measure** *p* **which is the closest** to a given belief function *b*

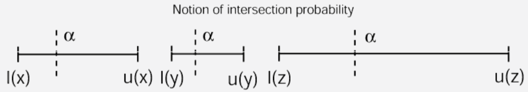


- two operators acting on bfs: **affine** combination and **Dempster's rule**
- elegant classification: two groups of transformations, **affine** and **epistemic** family

Affine family of transformations

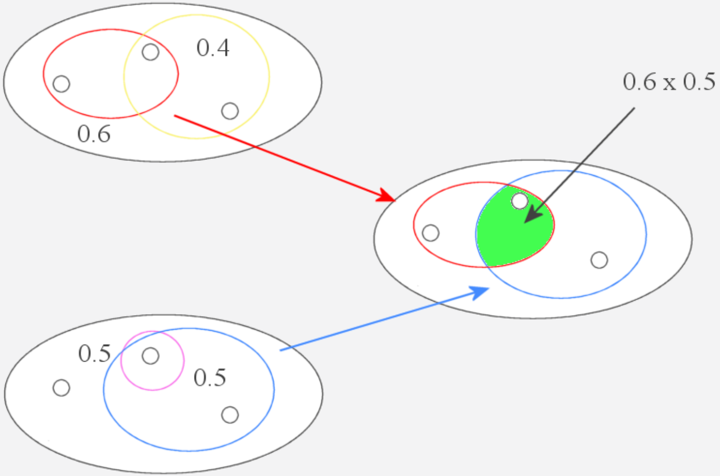
- **commute with affine combination of belief functions!**

- **pignistic function** [Smets] is the center of mass of all probabilities consistent with b
- **orthogonal projection** of b onto the probability simplex $\mathcal{P}$ [Cuzzolin]
- **intersection probability** [Cuzzolin]: assigns to each element the same fraction of the uncertainty given by a probability interval



Dempster's sum of belief functions

- originally proposed by A. Dempster



Epistemic family of transformations

Relative plausibility of singletons

- we seek transformations which **commute** with Dempster's rule [Cobb and Shenoy]

- plausibility function: $pl_b(A) = 1 - b(A^c)$
- **relative plausibility of singletons** $\tilde{pl}_b$ assigns to each element x its normalized plausibility
- proven to be **equivalent** to the original b.f. b when combined with a probability [Voorbraak]

$$b \oplus p = \tilde{pl}_b \oplus b \quad \forall p \in \mathcal{P}$$

Relative belief of singletons

- similar expression in which pl_b is replaced by b

$$\tilde{b}(x) \doteq \frac{b(x)}{\sum_{y \in \Theta} b(y)} = \frac{m(x)}{\sum_{y \in \Theta} m(y)}$$

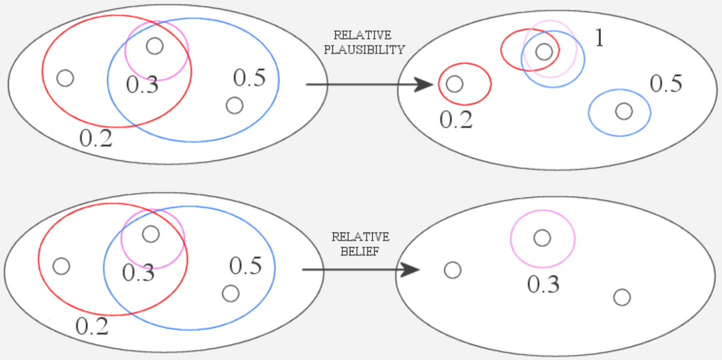
- assigns to each element its **normalized belief value**
- exists **if and only if** some mass is assigned to singletons

$$\sum_{x \in \Theta} m(x) \neq 0.$$

- this is true in general, i.e. **the set of belief measures for which $\tilde{b}$ does not is lower-dimensional**

Example of relative plausibility and belief

- example of the two transformations on a domain with 3 points



Semantics of relative plausibility

- **optimistic** estimate of support to x: $pl_b(x) = \sum_{A \supseteq \{x\}} m(A)$

- for each singleton $x \in \Theta$ the most *optimistic* hypothesis in which the mass of all $A \supseteq \{x\}$ focuses on x is considered, yielding $\{pl_b(x), x \in \Theta\}$;
- this assumption, however, is contradictory as it is supposed to hold for all singletons (many of which belong to the same higher-size events);
- nevertheless, the obtained values are normalized to yield a Bayesian belief function.

Dual semantics of relative belief

- **conservative** estimate of support to x
- for each singleton $x \in \Theta$ the most *pessimistic* hypothesis in which only the mass of $\{x\}$ itself actually focuses on x is considered, yielding $\{b(x) = m(x), x \in \Theta\}$;
- this assumption is also *contradictory*, as the mass of all higher-size events is not assigned to any singletons;
- the obtained values are again *normalized* to produce a Bayesian belief function

A strongly linked couple

- is it correct to interpret them as probabilistic "approximations"?
- thought of as "approximations" only for quasi-Bayesian belief functions

Theorem

For quasi-Bayesian b.f.s all Bayesian approximations converge:

$$\lim_{k_m \to 1} BetP[b] = \lim_{k_m \to 1} \tilde{pl}_b = \lim_{k_m \to 1} \tilde{b}.$$

- they are **not necessarily consistent** with the original b
- most correct interpretation in a **game theory** framework

A game-theoretical interpretation

- a utility function $u : \Theta \rightarrow \mathbb{R}^+$ on the elements of Θ
- belief value of x measure the minimal support to x, under the family of probabilities $\mathcal{P}[b]$ consistent with b

$$b(x) = \min_{p \in \mathcal{P}[b]} p(x).$$

- suppose an opponent can pick any consistent probability
- then **the peak of relative belief**

$$x_{maximin} = \arg\max_{x \in \Theta} \tilde{b}(x) = \arg\max_{x \in \Theta} \min_{p \in \mathcal{P}[b]} p(x).$$

represents the **best possible defensive strategy** aimed at maximizing the minimal utility of the possible outcomes

Properties of relative belief and $\oplus$

Proposition

*The relative plausibility transformation **commutes** with respect to Dempster's combination of plausibility functions, namely*

$$\tilde{pl}_b[b_1 \oplus b_2] = \tilde{pl}_b[b_1] \oplus \tilde{pl}_b[b_2].$$

*The relative plausibility of singletons $\tilde{pl}_b$ **represents perfectly** the corresponding belief function b when combined with any probability through Dempster's rule:*

$$\tilde{pl}_b \oplus p = b \oplus p$$

for each probability distribution $p \in \mathcal{P}$.

(Extended) Dempster's sum of plausibility functions

- **pseudo belief functions** $\varsigma : 2^\Theta \rightarrow \mathbb{R}$ are functions s.t.

$$\varsigma(A) = \sum_{B \subseteq A} m_\varsigma(B)$$

  where m_ς can be negative
- Dempster's sum can be applied to pseudo belief functions
- plausibility functions are themselves pseudo belief functions
- we can compute **Dempster's sums of plausibility functions**!

Dual properties

... of relative belief

Proposition

The relative belief transformation ***commutes*** *with respect to Dempster's combination of plausibility functions, namely*

$$\tilde{b}[pl_1 \oplus pl_2] = \tilde{b}[pl_1] \oplus \tilde{b}[pl_2].$$

The relative belief of singletons $\tilde{b}$ ***represents perfectly*** *the corresponding plausibility function* pl_b *when combined with any probability through (extended) Dempster's rule:*

$$\tilde{b} \oplus p = pl_b \oplus p$$

for each Bayesian belief function $p \in \mathcal{P}$*.*

Dual properties

... of relative belief

Proposition

*If pl_b is **idempotent** with respect to Dempster's rule, i.e. $pl_b \oplus pl_b = pl_b$, then $\tilde{b}[pl_b]$ is itself **idempotent**:*

$$\tilde{b}[pl_b] \oplus \tilde{b}[pl_b] = \tilde{b}[pl_b].$$

If $\exists x \in \Theta$ such that $b(x) > b(y)$ $\forall y \neq x, y \in \Theta$, then

$$\tilde{b}[pl_b^\infty](x) = 1, \quad \tilde{b}[pl_b^\infty](y) = 0 \; \forall y \neq x$$

where pl_b^∞ denotes the infinite limit of the combination of pl_b with itself.

Conclusions

... and further thoughts

- two families of probability transformations
- relate to affine and Dempster's operator respectively
- relative belief and relative plausibility of singletons form a **dual couple**
- not correct to interpret them as "approximations" of a belief function
- best defensive strategies in a game theory scenario
- **dual properties** with respect to the rule of combination