Boolean and matroidal independence in uncertainty theory

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The theory of evidence

- formulated as a theory of **subjective probability**;
- mathematical description of how a body of evidence affects one’s belief;
- contains standard probability as special case;
- knowledge state represented by **belief functions** instead of finite probabilities;
- Bayes’ rule is replaced by more general **Dempster’s rule**;
Belief and probability measures

- **Probability distribution**: $p : \Theta \rightarrow [0, 1]$ s.t.
  
  $p(\emptyset) = 0, \sum_{x \in \Theta} p(x) = 1, p(x) \geq 0 \ \forall x \in \Theta$

- **Probability measure** $p(A) = \sum_{x \in A} p(x)$

- **Basic belief assignment** $m : 2^\Theta \rightarrow [0, 1]$ such that
  
  $m(\emptyset) = 0, \sum_{A \subseteq \Theta} m(A) = 1, m(A) \geq 0 \ \forall A \subseteq \Theta$

- **Belief function** $b : 2^\Theta \rightarrow [0, 1]$: $b(A) = \sum_{B \subseteq A} m(B)$
Belief functions and multi-valued maps

- suppose you have a probability measure on a certain question $Q_1$;
- you want degrees of belief for a different question $Q_2$ ...
- having a one-to-many relation between answers to $Q_1$ and answers to $Q_2$; a **multi-valued map**;
- what you get is a belief function!
Dempster’s combination

**Definition**

The *orthogonal sum* or *Dempster’s sum* of two b.f.s $b_1, b_2$ on $\Theta$ is a new belief function $b_1 \oplus b_2$ on $\Theta$ with b.p.a.

$$m_{b_1 \oplus b_2}(A) = \frac{\sum_{B \cap C = A} m_{b_1}(B)m_{b_2}(C)}{\sum_{B \cap C \neq \emptyset} m_{b_1}(B)m_{b_2}(C)}.$$

When the denominator of the above equation is zero the two b.f.s are said to be *non-combinable*. 
Example

- $m_1(a_1) = 0.7$, $m_1(a_1, a_2) = 0.3$;

- $m_2(\Theta) = 0.1$, $m_2(a_2, a_3, a_4) = 0.9$;

- $m_1 \oplus m_2(a_1) = 0.19$,
  $m_1 \oplus m_2(a_2) = 0.73$,
  $m_1 \oplus m_2(a_1, a_2) = 0.08$. 
Explanation in terms of multi-valued maps

- Consider two multi-valued maps to the same domain $\Theta$;
- And two probabilities on $\Omega_1, \Omega_2$ inducing two different belief functions on $\Theta$;
- If the two pieces of evidence $P_1, P_2$ are independent we can build the product space:
  $$
  \Omega = \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 | \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \neq \emptyset \}
  $$
- But we have to remember to factor out empty intersections!
- What we get is the Dempster's combination of the two b.f.
Refining
between pairs of finite domains or frames

- consider two frames \( \Theta, \Theta' \);

**Definition**

A map \( \rho : 2^\Theta \to 2^{\Theta'} \) is a *refining* when it maps \( \Theta \) to a disjoint partition of \( \Theta' \)

- \( \Theta' \) **refinement** of \( \Theta \)
Family of compatible frames

Definition

In a **family of compatible frames** each finite collection of frames admits a common refinement (amongst other things)
Independence of frames
as Boolean sub-algebras

**Definition**

\( \Theta_1, \ldots, \Theta_n \) are *independent* [Shafer’76] (\( I\mathcal{F} \)) if

\[
\rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset
\]

(1)

whenever \( \emptyset \neq A_i \subset \Theta_i \) for \( \forall i \);
Independence of frames and Dempster’s rule

Proposition

$\Theta_1, \ldots, \Theta_n$ are independent iff all the possible collections of b.f.s $b_1, \ldots, b_n$ on $\Theta_1, \ldots, \Theta_n$ are combinable on their minimal refinement $\Theta_1 \otimes \cdots \otimes \Theta_n$

1. compare the condition of existence of Dempster’s sum ..

$$\Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \neq \emptyset, \quad (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$$

2. with independence of frames!

$$\rho_1(\theta_1) \cap \rho_2(\theta_2) \neq \emptyset, \quad (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2.$$
Basic notions

- generalize several originally distinct forms of independence introduced in different contexts;

**Definition**

A matroid $M = (E, \mathcal{I})$ is a pair formed by a ground set $E$, and a collection of independent sets $\mathcal{I} \subseteq 2^E$, such that:

1. $\emptyset \in \mathcal{I}$;
2. if $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$;
3. if $I_1$ and $I_2$ are in $\mathcal{I}$, and $|I_1| < |I_2|$, then there is an element $e$ of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

- the augmentation axiom 3. is the foundation of abstract independence;
Consider as an example the matrix

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 
\end{bmatrix}
$$

with column labels $E = \{1, 2, 3, 4, 5\}$;

the collection of independent sets in $E$ is $I = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$;

and it meets the augmentation axiom!
Families of frames are not matroids

**Theorem**

A family of compatible frames $\mathcal{F}$ endowed with Shafer’s independence $\mathcal{IF}$ is not a matroid.

**Proof.**

In fact, $\mathcal{IF}$ does not meet the augmentation axiom 3!

However, frames form a different algebraic structure in which independence can also be defined.
Families of frames as semi-modular lattices

- order relation: $\Theta_1 \leq \Theta_2$ iff $\Theta_1$ is a refinement of $\Theta_2$;
- $\sup$ – $>$ least upper bound, $\inf$ – $>$ greatest lower bound;

**Definition**

A *lattice* $L$ is a poset in which each *pair* of elements admits both $\inf$ and $\sup$.

**Proposition**

A *family of frames* is a semi-modular lattice with respect to the above order relation.
Families of frames as geometric lattices

atoms: elements just above zero;

**Definition**

$L$ is called *geometric* if it is semi-modular and each compact element of $L$ is a join of atoms: $\forall p \in L \exists a_1, \ldots, a_m \in A$ such that $p = \bigvee_i a_i$.

classical example: projective geometries $L(V)$

- compact elements: finite-dim subspaces;
- each finite-dim vector subspace is the span of a finite number of vectors (atoms of $L(V)$)

**Theorem**

*The partition lattice $L(\Theta)$ of any frame $\Theta$ is a geometric lattice.*
An analogy

- between independence of vector spaces and independence of frames:

\[ \rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset, \ \forall A_i \subset \Theta_i \]

\[ v_1 + \cdots + v_n \neq 0, \ \forall v_i \in V_i. \]

- consequence of families of frames and projective geometries sharing the structure of geometric lattice!

- but geometric lattices, again, are strictly related to matroids
The geometric lattice of flats

- basis: maximal independent set;
- rank $r(X)$ of a set $X$ is the size of a basis of $M|X$;
- closure of a set $X$:

$$cl(X) = \{ x \in E : r(X \cup x) = r(X) \}$$

- A flat $F$ of a matroid $M$ is a set which coincides with its closure: $F = cl(F)$.

**Proposition**

A lattice $L$ is geometric iff it is the lattice of flats of a matroid $M$. 
Independence of flats?
as a possible generalization of both

- we can abstract from the analogy seen before ...
- and come out with a more general definition of independence:

**Definition**

A collection $F_1, ..., F_n$ of flats of a matroid $\mathcal{M}$ is flat-independent (FI) if each possible selection of $n$ representatives of $F_1, ..., F_n$ is independent in $\mathcal{M}$, i.e.

$$\{f_1, ..., f_n\} \in \mathcal{I} \quad \forall f_1 \in F_1, ..., f_n \in F_n.$$  

- if formally explains the symmetry, but ..
Triviality!

- The matroid which has the frames as flats is **trivial**!

**Theorem**

The matroid whose flats are all the frames of a family of partitions $L(\Theta)$ is the trivial matroid

$$\mathcal{M} = (A, 2^A)$$

on the collection $A$ of atoms

- all partitions of cardinality $|\Theta| - 1$ are "independent" in this sense
- hence it cannot correspond to the independence of frames
we tried to reduce independence of frames to some sort of matroidal independence;

in fact, independence in the Boolean sense and matroidal independence are in opposition!

\( \{l_1, ..., l_n\} \) are \( \mathcal{I} \) if \( h(\bigvee_i l_i) = \sum_i h(l_i) \) where \( h(l) \) is the height of \( l \);

**Proposition**

\((A, \mathcal{I})\), where \( A \) is the set of atoms of a semi-modular lattice with initial element, is a matroid.

example: linear independence of vectors, i.e. the atoms of the projective geometry \( L(V) \);
The anti-matroid of independent binary frames
Mutual exclusivity of independence of frames and matroidal independence

Theorem

Pairs of binary partitions of a frame $\Theta$ (atoms of the lattice $L^*(\Theta)$) are independent as frames ($\mathcal{IF}$) if and only if they are not independent as elements of a matroid ($\mathcal{I}$).
Conclusions
and what we still do not know!

- independence of sources in the ToE is equivalent to independence of frames as Boolean sub-algebras;
- independence of frames does not produce a matroid;
- however, frames form both semi-modular and geometric lattice;
- this though does not fully explain the analogy with classical linear independence;
- in fact, independence of frames and matroidal indep are opposed to each other;
- more to learn on the general case, need for a more comprehensive definition!