Boolean and matroidal independence

in uncertainty theory

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Outline

The theory of evidence

- formulated as a theory of **subjective probability**;
- mathematical description of how a body of evidence affects one's belief;
- contains standard probability as special case;
- knowledge state represented by **belief functions** instead of finite probabilities;
- Bayes' rule is replaced by more general **Dempster's rule**;

Belief and probability measures

- probability distribution: $p : \Theta \rightarrow [0,1]$ s.t.

$$p(\emptyset) = 0, \sum_{x \in \Theta} p(x) = 1, p(x) \geq 0 \ \forall x \in \Theta$$

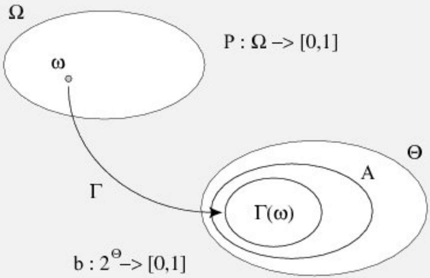
- probability measure $p(A) = \sum_{x \in A} p(x)$
- **Basic belief assignment** $m : 2^\Theta \rightarrow [0,1]$ such that

$$m(\emptyset) = 0, \sum_{A \subseteq \Theta} m(A) = 1, m(A) \geq 0 \ \forall A \subseteq \Theta$$

- **belief function** $b : 2^\Theta \rightarrow [0,1]$: $b(A) = \sum_{B \subseteq A} m(B)$

Belief functions and multi-valued maps

- suppose you have a probability measure on a certain question Q_1;
- you want degrees of belief for a different question Q_2 ...
- having a one-to-many relation between answers to Q_1 and answers to Q_2: a **multi-valued map**;
- what you get is a belief function!



Dempster's combination

Definition

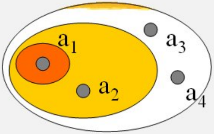
The *orthogonal sum* or *Dempster's sum* of two b.f.s b_1, b_2 on Θ is a new belief function $b_1 \oplus b_2$ on Θ with b.p.a.

$$m_{b_1 \oplus b_2}(A) = \frac{\sum_{B \cap C = A} m_{b_1}(B) m_{b_2}(C)}{\sum_{B \cap C \neq \emptyset} m_{b_1}(B) m_{b_2}(C)}.$$

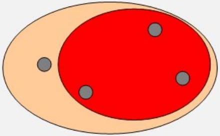
When the denominator of the above equation is zero the two b.f.s are said to be *non-combinable*.

Example

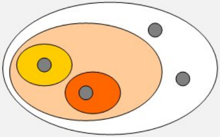
- $m_1(a_1) = 0.7,\ m_1(a_1, a_2) = 0.3;$



- $m_2(\Theta) = 0.1,\ m_2(a_2, a_3, a_4) = 0.9;$



- $m_1 \oplus m_2(a_1) = 0.19,$
  $m_1 \oplus m_2(a_2) = 0.73,$
  $m_1 \oplus m_2(a_1, a_2) = 0.08.$



Explanation in terms of multi-valued maps

- consider two multi-valued maps to the same domain Θ;
- and two probabilities on Ω_1, Ω_2 inducing two different belief functions on Θ;
- if the two pieces of evidence P_1, P_2 are independent we can build the product space:

$$\Omega = \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 | \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \neq \emptyset\}$$

- but we have to remember to factor out empty intersections!
- what we get is the Dempster's combination of the two b.f.

Refining

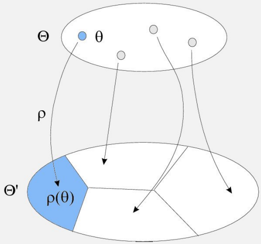
between pairs of finite domains or frames

- consider two frames Θ, Θ';

Definition

A map $\rho : 2^{\Theta} \rightarrow 2^{\Theta'}$ is a *refining* when it maps Θ to a disjoint partition of Θ'

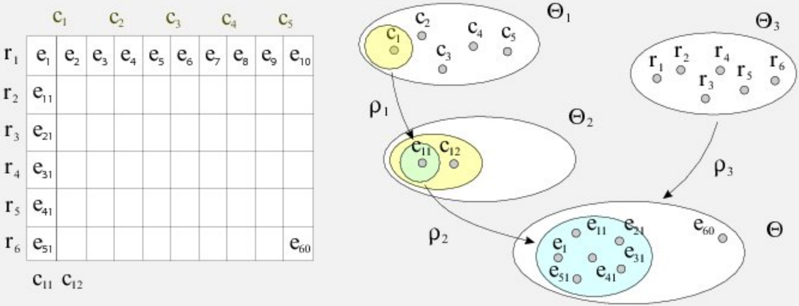
- Θ' **refinement** of Θ



Family of compatible frames

Definition

In a **family of compatible frames** each finite collection of frames admits a common refinement (amongst other things)



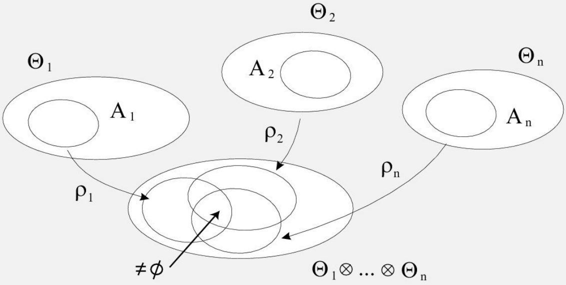
Independence of frames
as Boolean sub-algebras

Definition

$\Theta_1, ..., \Theta_n$ are *independent* [Shafer'76] ($\mathcal{IF}$) if

$$\rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset \tag{1}$$

whenever $\emptyset \neq A_i \subset \Theta_i$ for $\forall i$;



Independence of frames and Dempster's rule

Proposition

$\Theta_1, ..., \Theta_n$ are independent iff all the possible collections of b.f.s $b_1, ..., b_n$ on $\Theta_1, ..., \Theta_n$ are combinable on their minimal refinement $\Theta_1 \otimes \cdots \otimes \Theta_n$

- compare the condition of existence of Dempster's sum ..

$$\Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \neq \emptyset, \quad (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$$

- with independence of frames!

$$\rho_1(\theta_1) \cap \rho_2(\theta_2) \neq \emptyset, \quad (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2.$$

Basic notions

- generalize several originally distinct forms of independence introduced in different contexts;

Definition

A *matroid* $M = (E, \mathcal{I})$ is a pair formed by a *ground set* E, and a collection of *independent sets* $\mathcal{I} \subseteq 2^E$, such that:

1. $\emptyset \in \mathcal{I}$;
2. if $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$;
3. if I_1 and I_2 are in $\mathcal{I}$, and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

- the augmentation axiom 3. is the foundation of abstract independence;

Example: Vector matroids

- consider as an example the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

  with column labels $E = \{1, 2, 3, 4, 5\}$;
- the collection of independent sets in E is $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$;
- and it meets the augmentation axiom!

Families of frames are not matroids

Theorem

A family of compatible frames $\mathcal{F}$ endowed with Shafer's independence $\mathcal{IF}$ is not a matroid.

Proof.

In fact, $\mathcal{IF}$ does not meet the augmentation axiom 3! □

- however, frames form a different algebraic structure in which independence can also be defined

Families of frames as semi-modular lattices

- order relation: $\Theta_1 \leq \Theta_2$ iff Θ_1 is a refinement of Θ_2;
- sup $->$ least upper bound, inf $->$ greatest lower bound;

Definition

A *lattice L* is a poset in which each *pair* of elements admits both inf and sup.

Proposition

A family of frames is a semi-modular lattice with respect to the above order relation

Families of frames as geometric lattices

atoms: elements just above zero;

Definition

L is called *geometric* if it is semi-modular and

- each compact element of L is a join of atoms: $\forall p \in L$ $\exists a_1, \cdots, a_m \in A$ such that $p = \bigvee_i a_i$.

classical example: projective geometries $L(V)$

- compact elements: finite-dim subspaces;
- each finite-dim vector subspace is the span of a finite number of vectors (atoms of $L(V)$)

Theorem

The partition lattice $L(\Theta)$ of any frame Θ is a geometric lattice.

An analogy

- between independence of vector spaces and independence of frames:

$$\rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset, \ \forall A_i \subset \Theta_i$$

$$v_1 + \cdots + v_n \neq 0, \ \forall v_i \in V_i.$$

- consequence of families of frames and projective geometries sharing the structure of geometric lattice!
- but geometric lattices, again, are strictly related to matroids

The geometric lattice of flats

- basis: maximal independent set;
- *rank* $r(X)$ of a set X is the size of a basis of $M|X$;
- closure of a set X:

$$cl(X) = \{x \in E : r(X \cup x) = r(X)\}$$

- A *flat* F of a matroid M is a set which coincides with its closure: $F = cl(F)$.

Proposition

A lattice L is geometric iff it is the lattice of flats of a matroid M.

Independence of flats?

as a possible generalization of both

- we can abstract from the analogy seen before ...
- and come out with a more general definition of independence:

Definition

A collection $F_1, ..., F_n$ of flats of a matroid $\mathcal{M}$ is *flat-independent* ($\mathcal{FI}$) if each possible selection of n representatives of $F_1, ..., F_n$ is independent in $\mathcal{M}$, i.e.

$$\{f_1, ..., f_n\} \in \mathcal{I} \quad \forall f_1 \in F_1, ..., f_n \in F_n.$$

- if formally explains the symmetry, but ..

Triviality!

- the matroid which has the frames as flats is **trivial!**

Theorem

The matroid whose flats are all the frames of a family of partitions $L(\Theta)$ is the trivial matroid

$$\mathcal{M} = (A, 2^A)$$

on the collection A of atoms

- all partitions of cardinality $|\Theta| - 1$ are "independent" in this sense
- hence it cannot correspond to the independence of frames

Atom matroid

of a semimodular lattice

- we tried to reduce independence of frames to some sort of matroidal independence;
- in fact, independence in the Boolean sense and matroidal independence are in opposition!
- $\{l_1, ..., l_n\}$ are $\mathcal{I}$ if $h\left(\bigvee_i l_i\right) = \sum_i h(l_i)$ where $h(l)$ is the height of l;

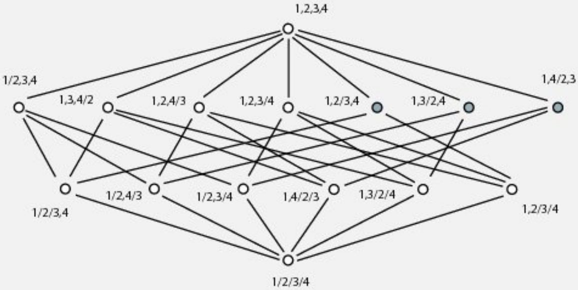
Proposition

$(A, \mathcal{I})$, where A is the set of atoms of a semi-modular lattice with initial element, is a matroid.

- example: linear independence of vectors, i.e. the atoms of the projective geometry $L(V)$;

The anti-matroid of independent binary frames

Mutual exclusivity of independence of frames and matroidal independence



Theorem

Pairs of binary partitions of a frame Θ (atoms of the lattice $L^*(\Theta)$) are independent as frames ($\mathcal{IF}$) if and only if they are not independent as elements of a matroid ($\mathcal{I}$).

Conclusions

and what we still do not know!

- independence of sources in the ToE is equivalent to independence of frames as Boolean sub-algebras;
- independence of frames does not produce a matroid;
- however, frames form both semi-modular and geometric lattice;
- this though does not fully explain the analogy with classical linear independence;
- in fact, independence of frames and matroidal indep are opposed to each other;
- more to learn on the general case, need for a more comprehensive definition!