An interpretation of consistent belief functions in terms of simplicial complexes

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Belief and probability measures

- basic belief assignment and **belief functions**
  - basic belief assignment $m : 2^{\Theta} \rightarrow [0, 1]$ such that
    \[
    m(\emptyset) = 0, \quad \sum_{A \subseteq \Theta} m(A) = 1, \quad m(A) \geq 0 \quad \forall A \subseteq \Theta
    \]
  - belief function $b : 2^{\Theta} \rightarrow [0, 1]$, 
    \[
    b(A) = \sum_{B \subseteq A} m(B)
    \]
- probability functions or Bayesian b.f.s: $m_b(A) = 0$, $|A| > 1$
- plausibility function: $pl_b(A) = 1 - b(A^c) = \sum_{B \cap A} m_b(B)$
Belief and probability measures

Example

- **focal elements**, sets with non-zero mass \((B_1, B_2)\);

\[
b(A) = m(B_1) + m(B_2);
\]
Possibility measures and distributions

- **Possibility measure**: a function $\text{Pos} : 2^\Theta \rightarrow [0, 1]$ such that $\text{Pos}(\emptyset) = 0$, $\text{Pos}(\Theta) = 1$ and
  \[ \text{Pos} \left( \bigcup_i A_i \right) = \sup_i \text{Pos}(A_i) \]
  for any family of subsets $\{A_i|A_i \in 2^\Theta, i \in I\}$;
- **possibility distribution** $\pi : \Theta \rightarrow [0, 1]$, $\pi(x) \doteq \text{Pos} \{x\}$;
- they meet
  \[ \text{Pos}(A) = \sup_{x \in A} \pi(x). \]
Consonant belief functions, i.e. b.f.s whose focal elements are nested;

Theorem

The plausibility function $pl_b$ associated with a b.f. $b$ is a possibility measure iff $b$ is consonant, with $\pi = \bar{pl}_b$. 
Consistent belief functions: all its focal elements have non-empty intersection;

consonant b.f.s are consistent, vice-versa does not hold;

**Theorem**

*The plausibility assignment $\overline{pl}_b$ associated with a b.f. $b$ is a possibility distribution iff the b.f. $b$ is consistent.*
Belief space

Geometric approach to uncertainty

- belief functions $b : 2^\Theta \rightarrow [0, 1]$ are completely specified by their $N - 2$ belief values $\{b(A), \emptyset \subset A \subset \Theta\}$, $N = 2^{|\Theta|}$.
- they can then be represented as a point of $\mathbb{R}^{N-2}$.

- belief functions form a simplex

$$B = Cl(b_A, \emptyset \subsetneq A \subsetneq \Theta);$$

- probabilities lie on a face of this simplex;
Example: the binary case

- each b.f. $b : 2^\Theta \rightarrow [0, 1]$ corresponds to a vector $[b(x) = m_b(x), b(y) = m_b(y)]$;
Definition

A \textit{simplicial complex} is a collection $\Sigma$ of simplices which satisfies the following properties:

1. if a simplex belongs to $\Sigma$, then all its faces of any dimension belong to $\Sigma$;
2. the intersection of any two simplices is a face of both the intersecting simplices.
Consonant complex

**Proposition**

The region $CO$ of consonant belief functions in the belief space is a simplicial complex.

- Consonant subspace

$$CO = \bigcup_{A_1 \subset \ldots \subset A_n} Cl(b_{A_1}, \ldots, b_{A_n});$$

- Each maximal simplex is associated with a maximal chain of sets $A_1 \subset \ldots \subset A_n$;
Properties of consistent belief functions

Properties of consistent belief functions
as sets of intersecting focal elements

- All possible lists of f.e.s associated with consistent b.f.s obviously correspond to all possible collections of intersecting events:

\[ \left\{ A_1, \ldots, A_m \subseteq \Theta : \bigcap_{i=1}^{m} A_i \neq \emptyset \right\} \]

- Maximal collections of events with non-empty intersection:

\[ \left\{ A \subseteq \Theta : A \ni x \right\} \]

- Consistent subspace:

\[ CS = \bigcup_{x \in \Theta} Cl(b_A, A \ni x). \]
The consistent complex

Consistent subspace in the ternary case

- All b.f.s \( b \in \mathcal{B}_3 \) are 6-dimensional vectors:
  \[
  [b(x), b(y), b(z), b(\{x, y\}), b(\{x, z\}), b(\{y, z\})]
  \]

- \( \text{Cl}(b_A : A \ni x) = \text{Cl}(b_x, b_{\{x,y\}}, b_{\{x,z\}}, b_\Theta) \), \( \text{Cl}(b_A : A \ni y) = \text{Cl}(b_y, b_{\{x,y\}}, b_{\{y,z\}}, b_\Theta) \), \( \text{Cl}(b_A : A \ni z) = \text{Cl}(b_z, b_{\{x,z\}}, b_{\{y,z\}}, b_\Theta) \).
The consistent complex

- the consonant subspace (i.e. the space of possibility measures) is a complex
- the consistent subspace (i.e. the space of possibility distributions) is also a complex!

**Theorem**

\( CS \) is a simplicial complex.
Simplices, complexes, duality

belief measures, probability measures

conditional subspace

possibility measures, assignments

singular subspace
The complex of singular belief functions

Singular belief functions
not combinable with all other b.f.s

Definition

The \textit{orthogonal sum} or \textit{Dempster’s sum} of two b.f.s $b_1, b_2$ on $\Theta$ is a new belief function $b_1 \oplus b_2$ on $\Theta$ with b.p.a.

$$m_{b_1 \oplus b_2}(A) = \frac{\sum_{B \cap C = A} m_{b_1}(B) m_{b_2}(C)}{\sum_{B \cap C \neq \emptyset} m_{b_1}(B) m_{b_2}(C)}$$

where $m_{b_i}$ denotes the b.p.a. associated with $b_i$.

- \textit{conditional subspace}
  \begin{align*}
  \langle b \rangle & \doteq \{ b \oplus b', \forall b' \in B : \exists b \oplus b' \} \\
  \text{\textbullet~conditional subspace}
  \end{align*}

- \textit{singular subspace}
  \begin{align*}
  \text{Sing} & \doteq \{ b \in B : \exists b' \in B : \forall b \oplus b' \} \\
  \text{\textbullet~singular subspace}
  \end{align*}
The complex of singular belief functions

**Singular and consistent complexes**

- Form of the singular subspace

\[ \text{Sing} = \bigcup_{x \in \Theta} \text{Cl}(b_A : A \subseteq \{x\}^c). \]

**Theorem**

*The singular subspace Sing is a simplicial complex.*

- Consistent and singular b.f.s are in 1-1 correspondence
Consistent coordinates of a belief function

Binary case

- each b.f. on \{x, y\} can be decomposed as

\[ b = \left( m(x) + \frac{m(\Theta)}{2} \right) \left( \frac{m(x)}{m(x) + \frac{m(\Theta)}{2}} b_x + \frac{m(\Theta)}{m(x) + \frac{m(\Theta)}{2}} b_\Theta \right) + \left( m(y) + \frac{m(\Theta)}{2} \right) \left( \frac{m(y)}{m(y) + \frac{m(\Theta)}{2}} b_y + \frac{m(\Theta)}{m(y) + \frac{m(\Theta)}{2}} b_\Theta \right) \]
Consistent coordinates of a belief function

**General case**

- **consistent projections:**

  \[
  b^x = \frac{1}{BetP[b](x)} \sum_{A \ni x} \frac{m(A)}{|A|} b_A, \quad x \in \Theta
  \]

- \( b \) lives in the \( n - 1 \) dimensional simplex \( \mathcal{P}^b \equiv Cl(b^x, x \in \Theta) \)

- convex decomposition of any belief function

  \[
  b = \sum_{A \subseteq \Theta} m(A) b_A = \sum_{x \in \Theta} \sum_{A \ni x} \frac{m(A)}{|A|} b_A = \\
  = \sum_{x \in \Theta} BetP[b](x) \frac{\sum_{A \ni x} \frac{m(A)}{|A|} b_A}{BetP[b](x)} = \sum_{x \in \Theta} BetP[b](x)b^x.
  \]

- the coefficients of this decomposition are the values of the pignistic function
Consistent coordinates of a belief function

Pictorial representation
Open issues

Approximations, geometry, and probability

- consistent and consonant approximations using $L_p$ norms;
- relation with inner approximations;
- extension to continuous frames of discernments [Smets];
- belief functions as iso-perimeters of convex bodies;
- relation with geometric probability;
- better understanding of the relation between uncertainty and combinatorics;
both belief and possibility measures can be described as points of a Cartesian space;

while belief measures form simplices, possibility measures/distributions form complexes;

dually, combinability and singularity relate to simplices/complexes;

the formalism can be applied to the approximation problem;

future developments to $L_p$ based approximations are on their way;

deeper relations between geometry and probability have the greatest interest (at least to me)