An interpretation of consistent belief functions

in terms of simplicial complexes

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Outline

Belief and probability measures

- basic belief assignment and **belief functions**
  - basic belief assignment $m : 2^\Theta \rightarrow [0,1]$ such that

$$m(\emptyset) = 0, \sum_{A \subseteq \Theta} m(A) = 1, m(A) \geq 0 \ \forall A \subseteq \Theta$$

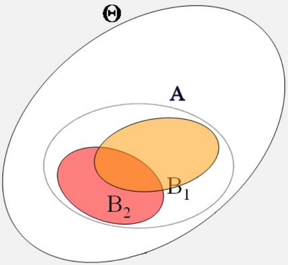
  - *belief function* $b : 2^\Theta \rightarrow [0,1]$,

$$b(A) = \sum_{B \subseteq A} m(B)$$

- probability functions or Bayesian b.f.s: $m_b(A) = 0, |A| > 1$
- plausibility function: $pl_b(A) = 1 - b(A^c) = \sum_{B \cap A} m_b(B)$

Belief and probability measures
Example

- **focal elements**, sets with non-zero mass (B_1, B_2);



- belief of A: $b(A) = m(B_1) + m(B_2)$;

Possibility measures and distributions

- **Possibility measure**: a function $Pos : 2^\Theta \to [0,1]$ such that $Pos(\emptyset) = 0$, $Pos(\Theta) = 1$ and

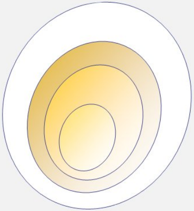
$$Pos\left(\bigcup_i A_i\right) = \sup_i Pos(A_i)$$

  for any family of subsets $\{A_i | A_i \in 2^\Theta, i \in I\}$;
- **possibility distribution** $\pi : \Theta \to [0,1]$, $\pi(x) \doteq Pos(\{x\})$;
- they meet

$$Pos(A) = \sup_{x \in A} \pi(x).$$

Consonant b.f.s as possibility measures

- **Consonant belief functions**, i.e. b.f.s whose focal elements are *nested*;

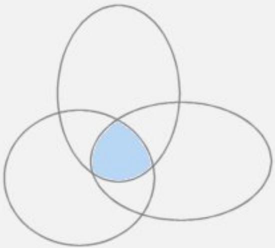


Theorem

The plausibility function pl_b associated with a b.f. b is a possibility measure iff b is consonant, with $\pi = \bar{pl}_b$.

Consistent b.f.s as possibility distributions

- **Consistent** belief functions: all its focal elements have non-empty intersection;



- consonant b.f.s are consistent, vice-versa does not hold;

Theorem

The plausibility assignment $\bar{pl}_b$ associated with a b.f. b is a possibility distribution iff the b.f. b is consistent.

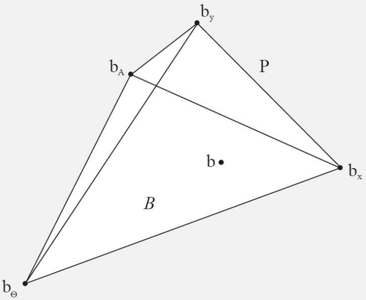
Geometric approach to uncertainty

- belief functions $b : 2^\Theta \rightarrow [0,1]$ are completely specified by their $N-2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$, $N \doteq 2^{|\Theta|}$
- they can then be represented as a point of $\mathbb{R}^{N-2}$

- belief functions form a simplex

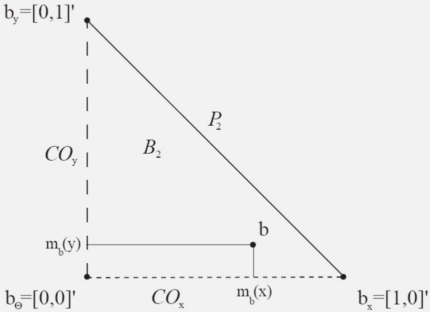
$$\mathcal{B} = Cl(b_A, \emptyset \subsetneq A \subsetneq \Theta);$$

- probabilities lie on a face of this simplex;



Example: the binary case

- each b.f. $b : 2^{\Theta_2} \rightarrow [0,1]$ corresponds to a vector $[b(x) = m_b(x), b(y) = m_b(y)]'$;

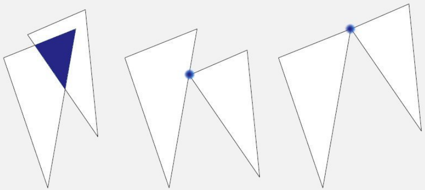


$$\mathcal{CO}_2 = \mathcal{CS}_2 = \mathcal{CO}_x \cup \mathcal{CO}_y = Cl(b_\Theta, b_x) \cup Cl(b_\Theta, b_y)$$

Simplicial complexes

Definition

A *simplicial complex* is a collection Σ of simplices which satisfies the following properties:
1. if a simplex belongs to Σ, then all its faces of any dimension belong to Σ;
2. the intersection of any two simplices is a face of both the intersecting simplices.



The consonant complex

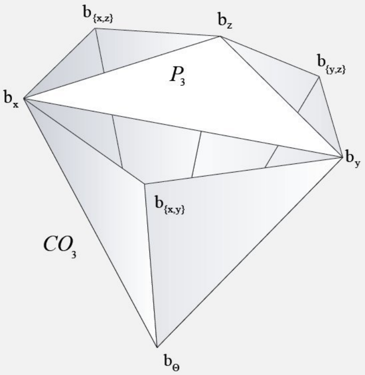
Proposition

The region $\mathcal{CO}$ of consonant belief functions in the belief space is a simplicial complex.

- consonant subspace

$$\mathcal{CO} = \bigcup_{A_1 \subset \dots \subset A_n} Cl(b_{A_1}, \dots, b_{A_n});$$

- each maximal simplex is associated with a maximal chain of sets $A_1 \subset \dots \subset A_n$;



Properties of consistent belief functions

as sets of intersecting focal elements

- All possible lists of f.e.s associated with consistent b.f.s obviously correspond to all possible collections of intersecting events:

$$\left\{ A_1, \ldots, A_m \subseteq \Theta : \bigcap_{i=1}^{m} A_i \neq \emptyset \right\}$$

- maximal collections of events with non-empty intersection:

$$\{ A \subseteq \Theta : A \ni x \}$$

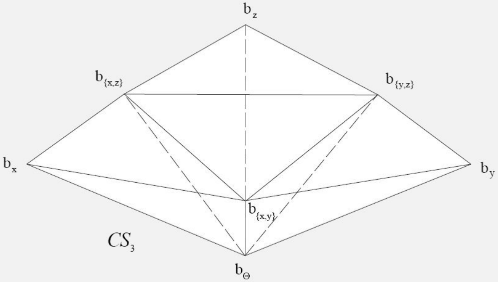
- **consistent subspace**:

$$\mathcal{CS} = \bigcup_{x \in \Theta} Cl(b_A, A \ni x).$$

Consistent subspace in the ternary case

- all b.f.s $b \in \mathcal{B}_3$ are 6-dimensional vectors:

$$[b(x), b(y), b(z), b(\{x,y\}), b(\{x,z\}), b(\{y,z\})]'$$



- $Cl(b_A : A \ni x) = Cl(b_x, b_{\{x,y\}}, b_{\{x,z\}}, b_\Theta)$, $Cl(b_A : A \ni y) = Cl(b_y, b_{\{x,y\}}, b_{\{y,z\}}, b_\Theta)$, $Cl(b_A : A \ni z) = Cl(b_z, b_{\{x,z\}}, b_{\{y,z\}}, b_\Theta)$.

The consistent complex

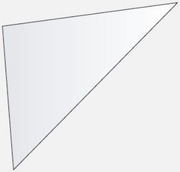
- the consonant subspace (i.e. the space of possibility *measures*) is a complex
- the consistent subspace (i.e. the space of possibility *distributions*) is also a complex!

Theorem

$\mathcal{CS}$ is a simplicial complex.

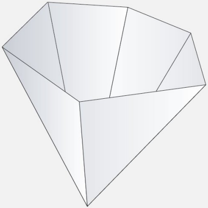
Simplices, complexes, duality

belief measures,
probability measures



conditional subspace

possibility measures,
assignments



singular
subspace

Singular belief functions

not combinable with all other b.f.s

Definition

The *orthogonal sum* or *Dempster's sum* of two b.f.s b_1, b_2 on Θ is a new belief function $b_1 \oplus b_2$ on Θ with b.p.a.

$$m_{b_1 \oplus b_2}(A) = \frac{\sum_{B \cap C = A} m_{b_1}(B)\, m_{b_2}(C)}{\sum_{B \cap C \neq \emptyset} m_{b_1}(B)\, m_{b_2}(C)}$$

where m_{b_i} denotes the b.p.a. associated with b_i.

- *conditional subspace*

$$\langle b \rangle \doteq \{ b \oplus b', \forall b' \in \mathcal{B} : \exists b \oplus b' \}$$

- *singular subspace*

$$Sing \doteq \{ b \in \mathcal{B} : \exists b' \in \mathcal{B} : \nexists b \oplus b' \}$$

Singular and consistent complexes

- form of the singular subspace

$$Sing = \bigcup_{x \in \Theta} Cl(b_A : A \subseteq \{x\}^c).$$

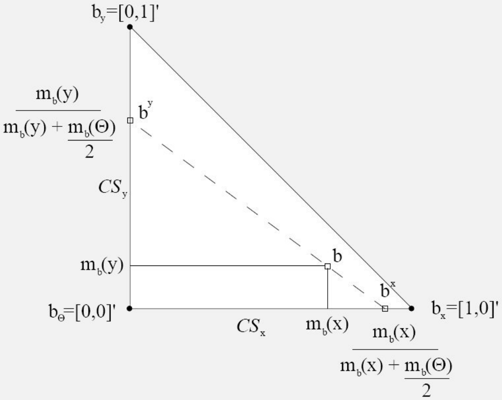
Theorem

The singular subspace Sing is a simplicial complex.

- consistent and singular b.f.s are in 1-1 correspondence

Binary case

- each b.f. on $\{x, y\}$ can be decomposed as



$$b = \left(m(x) + \frac{m(\Theta)}{2}\right)\left(\frac{m(x)}{m(x)+\frac{m(\Theta)}{2}} b_x + \frac{\frac{m(\Theta)}{2}}{m(x)+\frac{m(\Theta)}{2}} b_\Theta\right) + \left(m(y) + \frac{m(\Theta)}{2}\right)\left(\frac{m(y)}{m(y)+\frac{m(\Theta)}{2}})b_y + \frac{\frac{m(\Theta)}{2}}{m(y)+\frac{m(\Theta)}{2}} b_\Theta\right)$$

General case

- **consistent projections**:

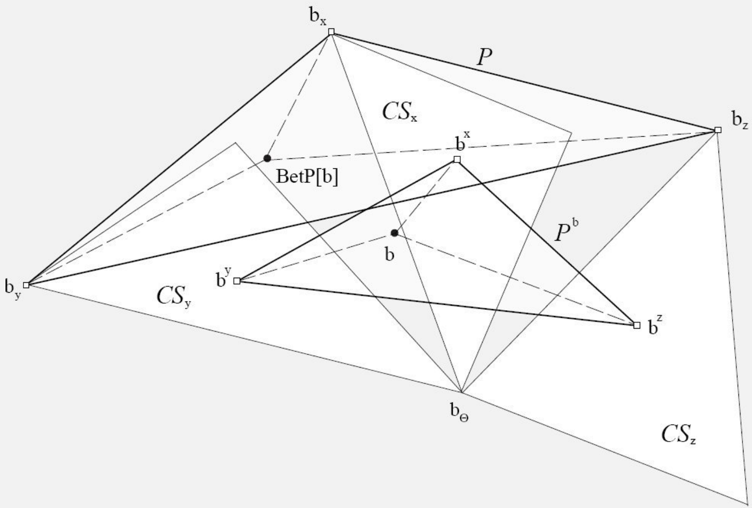
$$b^x \doteq \frac{1}{BetP[b](x)} \sum_{A \ni x} \frac{m(A)}{|A|} b_A, \qquad x \in \Theta$$

- b lives in the $n-1$ dimensional simplex $\mathcal{P}^b \doteq Cl(b^x, x \in \Theta)$
- convex decomposition of any belief function

$$b = \sum_{A \subseteq \Theta} m(A) b_A = \sum_{x \in \Theta} \sum_{A \ni x} \frac{m(A)}{|A|} b_A = \\ = \sum_{x \in \Theta} BetP[b](x) \frac{\sum_{A \ni x} \frac{m(A)}{|A|} b_A}{BetP[b](x)} = \sum_{x \in \Theta} BetP[b](x) b^x. \qquad (1)$$

- the coefficients of this decomposition are the values of the pignistic function

Pictorial representation



Open issues

Approximations, geometry, and probability

- consistent and consonant approximations using L_p norms;
- relation with inner approximations;
- extension to continuous frames of discernments [Smets];
- belief functions as iso-perimeters of convex bodies;
- relation with geometric probability;
- better understanding of the relation between uncertainty and combinatorics;

Conclusions

... and further thoughts

- both belief and possibility measures can be described as points of a Cartesian space;
- while belief measures form simplices, possibility measures/distributions form complexes;
- dually, combinability and singularity relate to simplices/complexes;
- the formalism can be applied to the approximation problem;
- future developments to L_p based approximations are on their way;
- deeper relations between geometry and probability have the greatest interest (at least to me)