Boolean and matroidal independence in uncertainty theory

Fabio Cuzzolin
INRIA Rhône-Alpes
655, avenue de l'Europe
38334 SAINT ISMIER CEDEX, France

Abstract

In this paper we discuss the nature of independence of sources in the theory of evidence from an algebraic point of view. Independence in Dempster's rule is equivalent to independence of frames $\mathcal{IF}$ as Boolean sub-algebras. $\mathcal{IF}$, however, cannot be explained neither in terms of classical matroidal independence, nor (even if finite families of frames form geometric lattices) as a cryptomorphic form of independence of flats on geometric lattices. Independence of frames is actually opposed to matroidal independence, giving a collection of frame the structure of "anti-matroid".

1 Introduction

The theory of evidence was born as a contribution to a mathematically rigorous description of the notion of subjective probability. In subjective probability, different observers (or "experts") of the same phenomenon possess in general different notions of what the decision space is. Mathematically, this translates into admitting the existence of several distinct representations of the decision space at different levels of refinement. Evidence will in general be available on some of those domains or *frames*, and will need to be "moved" to a common frame or "common refinement" in order to be fused. However, fusion through Dempster's rule (Dempster 1967; 1968; 1969) can surely take place (Cuzzolin 2005) only when those frames are *independent* (Shafer 1976). Combinability (in Dempster's sense) and independence of frames (in Shafer's formulation of the theory of evidence) are strictly intertwined.

The formal definition of evidence combination has been widely studied (Zadeh 1986; Yager 1987) in different mathematical frameworks (Smets 1990; Dubois & Prade 1992). An exhaustive review would be impossible here (Campos & de Souza 2005; Liu 2006; Murphy 2000; Carlson & Murphy 2005; Sentz & Ferson April 2002). In particular, some work has indeed been done on the issue of merging conflicting evidence (Deutsch-McLeish 1990; Josang, Daniel, & Vannoorenberghe 2003; Lefevre, Colot, & Vannoorenberghe 2002; Wierman 2001), specially in critical situations in which the latter is derived from dependent sources (Cattaneo 2003). On the other hand not much work has been done on the properties of the families of compatible frames (Shafer, Shenoy, & Mellouli 1987; Kohlas & Monney 1995; Cuzzolin 2005).

Here we build on the results obtained in (Cuzzolin 2005) to complete the algebraic analysis of families of frames and conduct a comparative study of the notion of independence, so central in the theory of evidence, in an algebraic setup. We first recall the fundamental result on the equivalence between independence of sources in Dempster's combination (Section 2) and independence of frames (Section 3). In this incarnation independence of sources can indeed be studied from an algebraic point of view, and compared with other classical forms of independence.

The classical paradigm of abstract independence is the notion of *matroid*. Matroids were introduced by Whitney in the Thirties (Whitney 1935) when he and other authors, among which van der Waerden (van der Waerden 1937), Mac Lane (Lane 1938), and Teichmuller (Teichmuller 1936) recognized that several different concepts of dependence (Harary & Tutte 1969; Beutelspacher & Rosenbaum 1998) in algebra have many properties in common with linear dependence of vectors. It is natural to conjecture that independence of frames may be a form of matroidal independence (Section 4): however, this is not the case (Theorem 1).

Matroids, however, are strictly related to another algebraic structure, that of *geometric lattice* which in turn admits its own particular notion of "independence". In Section 5 we indeed prove that finite families of frames are also geometric lattices (Theorem 2). As a lattice is geometric iff it is the lattice of all the closed sets or "flats" of some matroid, compatible frames can be seen as flats of some matroid. We therefore propose a new definition of *independence of flats* and discuss the possibility that it corresponds to evidential independence (Section 6).

In fact, as we argue in Section 7, the binary frames of a family are independent as Boolean algebras iff they are *not* independent as elements of the corresponding matroid. In a sense, then, we can say that collections of independent frames are "anti-matroids".

The overall picture is intriguing, and could in the future shed more light on the relationship between matroidal and Boolean independence in discrete mathematics, pointing out the necessity of a more general, comprehensive definition of this very important notion.

2 Independence of sources in Dempster's combination

Independence of sources is central in the theory of evidence, as it is required to fuse the evidence carried by two or more belief functions.

2.1 Belief functions and multi-valued maps

A *basic probability assignment* (b.p.a.) over a finite set or *frame* (Shafer 1976) Θ is a function $m : 2^\Theta \rightarrow [0, 1]$ on its power set $2^\Theta = \{A \subseteq \Theta\}$ such that

$$m(\emptyset) = 0, \quad \sum_{A \subseteq \Theta} m(A) = 1, \quad m(A) \geq 0 \; \forall A \subseteq \Theta.$$

The *belief function* (b.f.) $b : 2^\Theta \rightarrow [0, 1]$ associated with a basic probability assignment m on Θ is defined as

$$b(A) = \sum_{B \subseteq A} m(B).$$

The notion of belief function originally comes from a series of Dempster's works on upper and lower probabilities induced by multi-valued mappings (Dempster 1967; 1968; 1969). The following sketch of the nature of belief functions is abstracted from (Shafer 1990; Smets 1987).
Let us consider a problem in which we have probabilities for a question Q_1 and we want to derive a degree of belief for a related question Q_2. Let us call Ω and Θ the sets of possible answers of Q_1 and Q_2 respectively.
Given a probability measure P on Ω we want to derive a degree of belief $b(A)$ that $A \subseteq \Theta$ contains the correct response to Q_2 (see Figure 1).

If we call $\Gamma(\omega)$ the subset of answers to Q_2 compatible with $\omega \in \Omega$, each element ω tells us that the answer to Q_2 is somewhere in A whenever $\Gamma(\omega) \subseteq A$. The map $\Gamma : \Omega \rightarrow 2^\Theta$ (where 2^Θ denotes the collection of subsets of Θ) is called a *multi-valued mapping* from Ω to Θ.
The degree of belief $b(A)$ of an event $A \subseteq \Theta$ is then the total probability of all answers ω that satisfy the above condition, namely $b(A) = P(\{\omega | \Gamma(\omega) \subseteq A\})$. A basic probability as-

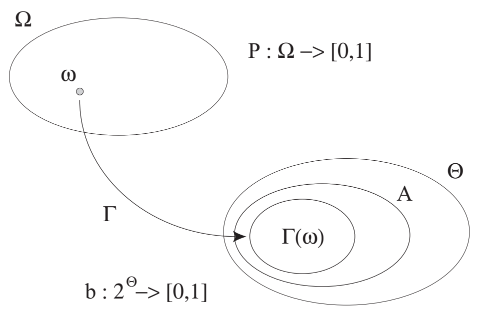

Figure 1: A probability measure P on Ω induces a belief function b on Θ whose values on the events A of Θ are given by $b(A) = \sum_{\omega \in \Omega : \Gamma(\omega) \subseteq A} P(\omega)$.

signment is then in fact an assignment of probability values (summing to one) to events of the frame, independently from their set-theoretic relation, as this assignment is induced by a multi-valued map.

This implies, for instance, that (unlike what happens with probability measures) the degree of belief of the disjoint union of two events A and B is *not* the sum of the associated degrees: in other words, a belief function *is not additive*: $b(A + B) \geq b(A) + b(B)$ (Shafer 1976).

2.2 Dempster's combination

In the ToE the available evidence is represented by a number of belief functions. To merge this evidence in order to make inferences on the response to the considered problem we then need to work out a mechanism for fusing two or more belief functions in a rational way.
The method originally proposed is based on an operator called Dempster's orthogonal sum.

Definition 1. *The* orthogonal sum *or* Dempster's sum *of two b.f.s* b_1, b_2 *on* Θ *is a new belief function* $b_1 \oplus b_2$ *on* Θ *with b.p.a.*

$$m_{b_1 \oplus b_2}(A) = \frac{\sum_{B \cap C = A} m_{b_1}(B) m_{b_2}(C)}{\sum_{B \cap C \neq \emptyset} m_{b_1}(B) m_{b_2}(C)}. \quad (1)$$

When the denominator of the above equation is zero the two b.f.s are said to be *non-combinable*.

The deep meaning of this operator can again be explained in terms of multi-valued maps.
Let us consider two multi-valued mappings Γ_1, Γ_2 inducing two belief functions over the same frame Θ, Ω_1 and Ω_2 their domains and P_1, P_2 the probability measures over Ω_1 and Ω_2 respectively. If we suppose that the items of evidence generating P_1 and P_2 *independent*, we are allowed to build the product space $(\Omega_1 \times \Omega_2, P_1 \times P_2)$: detecting two outcomes $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ will then tell us that the answer to Q_2 is somewhere in $\Gamma_1(\omega_1) \cap \Gamma_2(\omega_2)$.
However, if this intersection is empty the two pieces of evidence are in contradiction. We then need to condition the product measure $P_1 \times P_2$ over the set of pairs (ω_1, ω_2) whose images have non-empty intersection, namely

$$\begin{array}{l} \Omega = \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 | \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \neq \emptyset\}, \\ P = P_1 \times P_2|_\Omega, \qquad \Gamma(\omega_1, \omega_2) = \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2). \end{array} \quad (2)$$

The new b.f. b induced by (2) is indeed the Dempster's sum (1) of the pair of functions being combined.

3 Independence of sources and independence of frames

Dempster's mechanism for evidence combination is then intimately connected to the assumption that the domains on which the evidence is present (in the form of a probability measure) are independent. This relationship is mirrored by the notion of *independence of compatible frames*.

3.1 Families of compatible frames

Given two frames Θ and Θ', a map $\rho : 2^\Theta \rightarrow 2^{\Theta'}$ is a *refining* if $\forall A \subseteq \Theta \; \rho(A) = \cup_{\theta \in A} \rho(\{\theta\})$ and ρ maps the elements of Θ to a disjoint partition of Θ':

$$\rho(\{\theta\}) \cap \rho(\{\theta'\}) = \emptyset \;\; \forall \theta, \theta' \in \Theta, \quad \bigcup_{\theta \in \Theta} \rho(\{\theta\}) = \Theta'$$

(Figure 2). Θ' is called a *refinement* of Θ, Θ a *coarsening* of Θ'. Shafer calls a structured collection of frames a *family of compatible frames of discernment* ((Shafer 1976), pages 121-125). In particular, in such a family every pair of frames has a common refinement, i.e. a frame which is a refinement of both. If $\Theta_1, ..., \Theta_n$ belong to a family of compatible frames $\mathcal{F}$ then there exists a *unique* common refinement $\Theta \in \mathcal{F}$ of them s.t. $\forall \theta \in \Theta \; \exists \theta_i \in \Theta_i \; \forall i = 1, ..., n$ for which

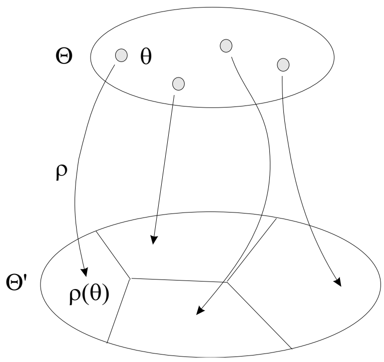

Figure 2: Refining.

$$\{\theta\} = \rho_1(\{\theta_1\}) \cap \cdots \cap \rho_n(\{\theta_n\}) \quad (3)$$

where ρ_i denotes the refining between Θ_i and Θ. This unique frame is called the *minimal refinement* $\Theta_1 \otimes \cdots \otimes \Theta_n$ of $\Theta_1, ..., \Theta_n$.

Example Let us consider a simple example drawn from image processing. We want to find out the position of a target point in an image. We can then pose the problem on a frame $\Theta_1 = \{c_1, ..., c_5\}$ obtained by partitioning the column range of the image into 5 intervals. The set of columns can be also partitioned into 10 intervals, yielding a new frame $\Theta_2 = \{c_{11}, c_{12}, ..., c_{51}, c_{52}\}$. On the other side, the row range can also be divided in, say, 6 intervals $\Theta_3 = \{r_1, ..., r_6\}$. All those frames are clearly related to the location of the target: as Figure 3 suggests, they all belong to a family of compatible frames, with the

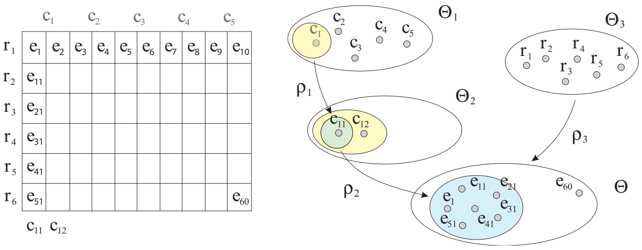

Figure 3: A family of compatible frames. Different discrete quantizations of row and column ranges of an image have a common refinement, the set of cells shown on the left. The refinings ρ_1, ρ_2, ρ_3 between those frames appear to the right.

collection of cells $\Theta = \{e_1, ..., e_{60}\}$ depicted in Figure 3-left as common refinement. Figure 3-right shows the refinings between them where, for instance, $\rho_1(c_1) = \{c_{11}, c_{12}\}$, $\rho_2(c_{11}) = \{e_1, e_{11}, e_{21}, e_{31}, e_{41}, e_{51}\}$, etcetera.
It is easy to verify that Θ meets condition (3) for the frames Θ_2, Θ_3 as, for example, $\{e_{41}\} = \rho_2(c_{11}) \cap \rho_3(r_4)$ i.e. Θ is the minimal refinement $\Theta_2 \otimes \Theta_3$ of Θ_2, Θ_3.

3.2 Independence of frames as Boolean sub-algebras

Now, let $\Theta_1, ..., \Theta_n$ be elements of a family of compatible frames, and $\rho_i : \Theta_i \to 2^{\Theta_1 \otimes \cdots \otimes \Theta_n}$ the corresponding refinings to their minimal refinement. $\Theta_1, ..., \Theta_n$ are *independent* (Shafer 1976) ($\mathcal{IF}$) if

$$\rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset \quad (4)$$

whenever $\emptyset \neq A_i \subset \Theta_i$ for $i = 1, ..., n$ (Figure 4). In

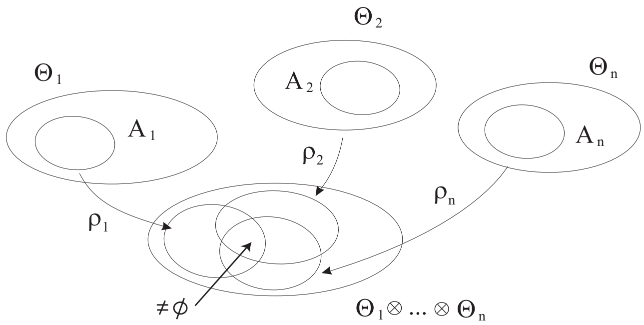

Figure 4: Independence of frames.

particular, if some Θ_j is a coarsening of some other frame Θ_i then $\Theta_1, ..., \Theta_n$ are *not* $\mathcal{IF}$.
A condition equivalent to (4) is (Cuzzolin 2005)

$$\Theta_1 \otimes \cdots \otimes \Theta_n = \Theta_1 \times \cdots \times \Theta_n \quad (5)$$

i.e. their minimal refinement is their Cartesian product.
Incidentally, the notion of independence of frames is a cryptomorphic version of that of independence of Boolean sub-algebras (Sikorski 1964). A collection of compatible frames $\Theta_1, ..., \Theta_n$ corresponds to a set of Boolean sub-algebras $2^{\Theta_1}, ..., 2^{\Theta_n}$ of the power set $2^{\Theta_1 \otimes \cdots \otimes \Theta_n}$ of their minimal refinement $\Theta_1 \otimes \cdots \otimes \Theta_n$. A set of sub-algebras $X_1, ..., X_n$ of a Boolean algebra $\mathcal{B}$ is *independent* if

$$\cap A_i \neq \wedge \quad (6)$$

$\forall \; A_i \in X_i$, where $\wedge \doteq \cap \mathcal{B}$ is the initial element of $\mathcal{B}$. For a collection of compatible frames (6) reads as (4).

3.3 Independence of frames and Dempster's rule

On the other side, independence of frames is strictly related to Dempster's combination (Cuzzolin 2005).

Proposition 1. *Let $\Theta_1, ..., \Theta_n$ be a set of compatible frames. Then they are independent iff all the possible collections of b.f.s $b_1, ..., b_n$ defined respectively on $\Theta_1, ..., \Theta_n$ are combinable on their minimal refinement $\Theta_1 \otimes \cdots \otimes \Theta_n$.*

Independence of frames and independence of sources (which is at the root of Dempster's combination) are in fact equivalent. This is not at all surprising when we compare the condition under which Dempster's sum is well defined (Equation 2)

$$\Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \neq \emptyset, \quad (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$$

with independence of frames

$$\rho_1(A_1) \cap \rho_2(A_2) \neq \emptyset, \quad A_1 \subset \Theta_1, A_2 \subset \Theta_2$$

which reduces to

$$\rho_1(\theta_1) \cap \rho_2(\theta_2) \neq \emptyset, \quad (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2.$$

See (Cuzzolin 2005) for a formal proof of Proposition 1. On this result an algebraic study of independence as introduced in the theory of evidence can be built.

4 Matroids

The classical paradigm of abstract independence is the notion of *matroid*. Matroids were introduced by Whitney in the Thirties (Whitney 1935) when he and other authors, among which van der Waerden (van der Waerden 1937), Mac Lane (Lane 1938), and Teichmuller (Teichmuller 1936) recognized that several different concepts of dependence (Harary & Tutte 1969; Beutelspacher & Rosenbaum 1998) in algebra had many properties in common with linear dependence of vectors. We will briefly introduce the basic notions (Oxley 1992) to later discuss how matroidal independence relates to independence of frames in the theory of evidence.

Definition 2. *A* matroid $M = (E, \mathcal{I})$ *is a pair formed by a* ground set E, *and a collection of* independent sets $\mathcal{I} \subseteq 2^E$, *such that:*

1. $\emptyset \in \mathcal{I}$;
2. *if* $I \in \mathcal{I}$ *and* $I' \subseteq I$ *then* $I' \in \mathcal{I}$;
3. *if* I_1 *and* I_2 *are in* $\mathcal{I}$, *and* $|I_1| < |I_2|$, *then there is an element* e *of* $I_2 - I_1$ *such that* $I_1 \cup e \in \mathcal{I}$.

Condition 3. is called *augmentation* axiom, and is the foundation of the notion of abstract independence in matroid theory. The name "matroid" was coined by Whitney (Whitney 1935) because of a fundamental class of matroids which arise from matrices.

Example: Vector matroids The collection of columns of a matrix together with the collection of linearly independent (in the ordinary sense) sets of columns form a matroid, called *vector matroid*. Consider as an example the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

with column labels $E = \{1, 2, 3, 4, 5\}$. Obviously the collection of independent sets in E is $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$. It is interesting to see that linearly independent vectors in a vector space actually satisfy the augmentation axiom 3. of Definition 2. Let I_1 and I_2 be linearly independent subsets such that $|I_1| < |I_2|$. Let W be the subspace spanned by $I_1 \cup I_2$. Then $\dim W$ is at least $|I_2|$ (as I_2, the largest of the two collections, is linearly independent). Now suppose that $I_1 \cup e$ is linearly dependent for all $e \in I_2 \setminus I_1$. Then W is contained in the span of I_1, thus $|I_2| \leq \dim W \leq |I_1| < |I_2|$ which is a contradiction.

Example: Graph matroids Another classical example of matroid arises from graph theory. Every finite graph G is associated with a matroid as follows: take as E the set of all edges in G and consider a set of edges independent if and only if it does not contain a simple cycle. Such an edge set is called a "forest" in graph theory. This is called the graphic matroid of G. Non-independent sets are called "circuits" in matroid theory, a terminology borrowed from graph theory.

4.1 Families of frames are not matroids

As matroidal independence formalizes several different notions of independence introduced in various fields of mathematics, we may wonder whether the notion of independence of frames (so important in the theory of evidence) could also be reduced to that of matroid. Matrices with linear independence, graphs with cycle independence are matroids. It makes then sense to conjecture that for each family $\mathcal{F}$ of compatible frames, $(\mathcal{F}, \mathcal{IF})$ is also a matroid. This would imply that independence of Boolean sub-algebras is itself a form of independence in matroidal sense. However, this is not the case.

Theorem 1. *A family of compatible frames* $\mathcal{F}$ *endowed with Shafer's independence* $\mathcal{IF}$ *is* not *a matroid.*

Proof. In fact, $\mathcal{IF}$ does not meet the augmentation axiom 3. of Definition 2. Consider two independent frames $I = \{\Theta_1, \Theta_2\}$. If we pick another arbitrary frame in the family Θ_3, the collection $I' = \{\Theta_3\}$ is trivially $\mathcal{IF}$. Suppose $\Theta_3 \neq \Theta_1, \Theta_2$. Then, since $|I| > |I'|$, by augmentation we can form a new pair of independent frames by adding any of Θ_1, Θ_2 to Θ_3. But it is easy to find a counterexample, for instance by picking $\Theta_3 = \Theta_1 \oplus \Theta_2$. As Θ_1 is a coarsening of $\Theta_1 \oplus \Theta_2$ this new pair is not $\mathcal{IF}$. ☐

Independence of Boolean sub-algebras is then not independence in matroidal sense. Matroids however are strictly related to the algebraic structure of *geometric* lattice, on which they induce a different definition of independence. As families of frames are indeed geometric lattices, we are then led to compare the latter with Boolean-theoretic $\mathcal{IF}$.

5 Families of frames as geometric lattices

The algebraic structure of families of compatible frames has already been studied in recent times (Kohlas & Monney 1995). In particular, it has been proven that they possess the algebraic structure of lattice: in particular, they belong to the class of *semi-modular* lattices (Cuzzolin 2005).

5.1 Lattices

A *partially ordered set* or *poset* is a set P together with a binary relation $\leq$ such that, for all x, y, z in P the following condition holds:

1. $x \leq x$;
2. if $x \leq y$ and $y \leq x$ then $x = y$;
3. if $x \leq y$ and $y \leq z$ then $x \leq z$.

In a poset we say that x "covers" y ($x \succ y$) if $x \geq y$ and there is no intermediate element in the chain linking them. A classical example is the power set 2^Θ of a set Θ together with the set-theoretic inclusion relation $\subset$.
Given two elements $x, y \in P$ of a poset P their *least upper bound* $\sup_P(x,y) = x \vee y$ or "join" is the smallest element of P that is bigger than both x and y, while their *greatest lower bound* $\inf_P(x,y) = x \wedge y$ is the biggest element of P that is smaller than both x and y. In the case of $L = (2^\Theta, \subset)$ "sup" is the usual set-theoretic union, $A \vee B = A \cup B$, while "inf" is the usual intersection $A \wedge B = A \cap B$. Not every pair of elements of a poset, though, admits inf and/or sup.

Definition 3. *A* lattice *L is a poset in which each* pair *of elements admits both* inf *and* sup.

When each *arbitrary* (even not finite) collection of elements of L admits both inf and sup, L is said *complete*. In this case there exist $\mathbf{0} \equiv \wedge L$, $\mathbf{1} \equiv \vee L$ called respectively *initial* and *final* element of L.
The elements of L covering $\mathbf{0}$ are called *atoms* of L: $a \succ \mathbf{0}$. The *height* $h(x)$ of an element x in L is the length of the maximal chain from $\mathbf{0}$ to x. In the case of the power set 2^Θ, the height of a subset $A \in 2^\Theta$ is simply its cardinality $|A|$. 2^Θ is complete, with $\mathbf{0} = \emptyset$ and $\mathbf{1} = \{\Theta\}$.

5.2 Families of frames as semi-modular lattices

Now, we can introduce in a family of frames the following order relation:

$$\Theta_1 \leq \Theta_2 \Leftrightarrow \exists \rho : 2^{\Theta_2} \to 2^{\Theta_1} \; refining \tag{7}$$

i.e. Θ_1 is smaller than Θ_2 iff Θ_1 is a refinement of Θ_2. Then (Cuzzolin 2005)

Proposition 2. *A family of frames is a lattice with respect to the order relation (7).*

In particular,

Definition 4. *A lattice L is* semi-modular *if for each pair x, y of elements of L, $x \succ x \wedge y$ implies $x \vee y \succ y$.*

We recently proved that (Cuzzolin 2007)

Proposition 3. *A family of frames is a semi-modular lattice with respect to the order relation (7).*

A different, lattice-theoretic notion of independence can be introduced on a semi-modular lattice. The relation between the latter and $\mathcal{IF}$ has been studied in (Cuzzolin 2007). Here we make use of Proposition 3 only to introduce a different algebraic structure on families of compatible frames.

5.3 The lattice of frames as a geometric lattice

The reader will be familiar with the notion of compactness: The latter can be given an abstract definition in terms of joins of a lattice.

Definition 5. *An element p of a lattice L is called* compact *iff if there exists a subset $S \subset L$ s.t. $p \leq \vee S$, then there exists a* finite *subset $F \subset S$ s.t. $p \leq \vee F$.*

Definition 6. *A lattice L is called* algebraic *if:*

- *L is complete (admits $\mathbf{0}$ and $\mathbf{1}$);*
- *each element p of L is a join of compact elements.*

L is called geometric *if:*

- *it is algebraic;*
- *it is upper semi-modular;*
- *each compact element of L is a join of atoms: $\forall p \in L$ $\exists a_1, \cdots, a_m \in A$ such that $p = \bigvee_i a_i$.*

Example: projective geometries The name "geometric" lattices comes in fact from the familiar case of "projective geometries", i.e. collections $L(V)$ of vector subspaces of a vector space V. Projective geometries are complete lattices, their compact elements being the finite-dimensional subspaces of V. Each finite dimensional subspace is a span of one-dimensional subspaces (vectors).
If a complete lattice L is finite, all its elements are joins of a finite number of atoms. In this case, geometricity reduces to semi-modularity.

As $\mathcal{IF}$ involves only partitions of $\Theta_1 \otimes \cdots \otimes \Theta_n$, we can conduct our analysis on the partition lattice $L(\Theta) \doteq (P(\Theta), \leq)$ associated with the set $P(\Theta)$ of all partitions of a given frame Θ. But $L(\Theta)$ is a complete finite semi-modular lattice, so that

Theorem 2. *$L(\Theta)$ is a geometric lattice.*

Compatible frames have then something to do with linear subspaces. We will see in the rest of the paper what this implies for the notion of independence.

6 $\mathcal{IF}$ and independence of flats

6.1 An analogy

An intriguing similarity indeed emerges between independence of frames and independence of vector subspaces in a projective geometry, evident from the following diagram (recalling Equations (4) and (5)):

$$\begin{gathered} \rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset, \; \forall A_i \subset \Theta_i \\ v_1 + \cdots + v_n \neq 0, \; \forall v_i \in V_i. \end{gathered} \tag{8}$$

While a number of compatible frames $\Theta_1, ..., \Theta_n$ are $\mathcal{IF}$ iff each choice of their representatives $A_i \in 2^{\Theta_i}$ has non-empty intersection, a collection of vectors subspaces $V_1, ..., V_n$ is "independent" iff for each choice of vectors $v_i \in V_i$ their sum is non-zero. These two relations defined in apparently very different contexts can be formally obtained from each other under the following correspondence of quantities and operators:

$$v_i \leftrightarrow A_i, \;\; V_i \leftrightarrow 2^{\Theta_i}, \;\; + \leftrightarrow \cap, \;\; 0 \leftrightarrow \emptyset, \;\; \otimes \leftrightarrow span.$$

This analogy is in fact a consequence of families of frames and projective geometries sharing the structure of geometric lattice. Let us see how, and what insight this gives us in the perspective of understanding the relation between matroidal and evidential (Boolean) independence.

6.2 The geometric lattice of flats

The reason is geometric lattices are strictly related to matroids. The latter are completely specified by the list of the *maximal* independent sets, i.e. sets I of $\mathcal{I}$ such that $I \cup e \notin \mathcal{I}$ for any element e of E. A maximal independent set in M is called a *basis* of M. All bases B_i of a matroid have the same cardinality, $|B_1| = |B_2|$. We know this from linear algebra, but it remains true in this more abstract setting.

Now, let $M = (E, \mathcal{I})$ be a matroid and suppose $X \subset E$ be a subset of the ground set. The pair $(X, \mathcal{I}|X)$ with

$$\mathcal{I}|X \doteq \{I \cap X, I \in \mathcal{I}\}$$

is still a matroid. We call it the *restriction* of M to X.

Definition 7. *The* rank $r(X)$ *of a set X is the size of a basis of $M|X$.*

You can prove that $X \subset E$ is independent iff $|X| = r(X)$. The function $cl : 2^E \to 2^E$ defined for all $X \subseteq E$ by

$$cl(X) = \{x \in E : r(X \cup x) = r(X)\}$$

i.e. the set of elements of E which leave the rank unchanged when joined to X is called *closure operator*. If M is a matroid and $X \subseteq E$, we call $cl(X)$ the *closure* or *span* of X. Again, in a vector space the span of a collection $X = \{v_1, ..., v_m\}$ is the set of all vectors which lie in the space generated by X.

Definition 8. *A* flat *F of a matroid M is a set which coincides with its closure: $F = cl(F)$.*

The set of flats of a matroid, ordered by inclusion, forms a geometric lattice (see Birkhoff (Birkhoff 1935), Dilworth (Dilworth 1944), and Crapo and Rota (Crapo & Rota 1970)). In fact, the reverse implication holds too (Oxley 1992).

Proposition 4. *A lattice L is geometric* iff *it is the lattice of flats of a matroid M.*

The bottom line of the proof is that the matroid $M = (E, \mathcal{I})$ which corresponds to L has as ground set the set of atoms of L, $E = A$, and as rank function $r(X) = h(\vee_{x \in X} x)$ for each collection of atoms $X \subset A$.

6.3 Independence of flats

Now, Proposition 4 and Theorem 2 tells us that, as they form geometric lattices, vector subspaces and compatible frames of discernment both form the lattice of flats of a matroid. In the case of $L(V)$, the matroid of interest is the set of all vectors of a vector space provided with the usual linear independence relation.

It is then natural to try and abstract from Equation (8) and come up with an “independence” relation for flats of a matroid (or equivalently, for elements of a geometric lattice).

Definition 9. *A collection $F_1, ..., F_n$ of flats of a matroid $\mathcal{M}$ is* flat-independent *($\mathcal{FI}$) if each possible selection of n representatives of $F_1, ..., F_n$ is independent in $\mathcal{M}$, i.e.*

$$\{f_1, ..., f_n\} \in \mathcal{I} \quad \forall f_1 \in F_1, ..., f_n \in F_n.$$

Clearly, $\mathcal{FI}$ formally fits well both independence of vectors and independence of frames, as illustrated by Equation (8). If we could prove that $\mathcal{IF}$ is indeed a form of flat-independence $\mathcal{FI}$ for some matroid defined on frames, we would finally have an explanation of the relation between independence of frames (as Boolean sub-algebras) and independence in matroids.

6.4 Triviality of the associated matroid

In order to do this, we need to check what is the matroid $M(\Theta) = (E, \mathcal{I})$ whose flats correspond to the geometric lattice of frames $L(\Theta)$ of a finite family, and whether the associated independence relation $\mathcal{I}$ is related to independence of frames in the evidential formalism. Unfortunately,

Theorem 3. *The matroid whose flats are all the frames of a family of partitions $L(\Theta)$ is the trivial matroid*

$$\mathcal{M} = (A, 2^A)$$

on the collection of atoms (frames of cardinality $|\Theta| - 1$) of $L(\Theta)$, for which each collection of frames of cardinality $|\Theta| - 1$ is independent.

Proof. The ground set of the matroid is the set of atoms A of $L(\Theta)$, i.e. the set of all partitions of Θ of cardinality $|\Theta| - 1$. Its rank, according to the proof of Proposition 4, is

$$r(X = \{\Theta_1, ..., \Theta_n)) = h\Big(\bigvee_{x \in X} x\Big) =$$
$$= h(\oplus_{i=1}^n \Theta_i) \doteq |\Theta| - |\oplus_{i=1}^n \Theta_i|.$$

To find $\mathcal{I}$ we need to find the sets such that $|X| = r(X)$, i.e.

$$n = |\Theta| - |\oplus_{i=1}^n \Theta_i|$$

or

$$|\oplus_{i=1}^n \Theta_i| = |\Theta| - n.$$

But it is easy to see by induction on n that this is true for any collection of atoms in $L(\Theta)$.

For $n = 2$ by semi-modularity (Definition 4), since $\Theta_1, \Theta_2 \succ \Theta = \Theta_1 \wedge \Theta_2$ we have that

$$\Theta_1 \vee \Theta_2 = \Theta_1 \oplus \Theta_2 \succ \Theta_1, \Theta_2$$

i.e. $|\Theta_1 \oplus \Theta_2| = |\Theta_1| - 1 = |\Theta| - 2 = |\Theta| - 2$.

In the induction step we suppose that

$$|\oplus_{i=1}^{n-1} \Theta_i| = |\Theta| - (n-1).$$

But again, since $\Theta_n \succ \Theta = (\oplus_{i=1}^{n-1} \Theta_i) \wedge \Theta_n$,

$$\vee_i \Theta_i = \oplus_{i=1}^n \Theta_i \succ \oplus_{i=1}^{n-1} \Theta_i$$

i.e. $|\oplus_{i=1}^n \Theta_i| = |\oplus_{i=1}^{n-1} \Theta_i| - 1 = |\Theta| - n$. □

7 Independence of frames opposed to matroidal independence

So far, following the intuition provided by the shared lattice structure of $L(\Theta)$ and $L(V)$ we tried to reduce independence of frames to some sort of matroidal independence. Our efforts have been frustrated though, as independence of frames (or equivalently independence of sources or again Boolean independence) cannot be explained nor as independence of elements of a matroid (Theorem 1), neither as independence of flats of a matroid (Theorem 3).

We conclude this paper by showing that independence of sources is in fact *in opposition* to matroidal independence. Let us consider the following relation on the elements of a semi-modular lattice: $\{l_1, ..., l_n\}$ are $\mathcal{I}$ if

$$h\Big(\bigvee_i l_i\Big) = \sum_i h(l_i) \qquad (9)$$

where $h(l)$ is the height of l, i.e. the length of the shortest chain from l to $\mathbf{0}$. It is well known that (Szasz 1963)

Proposition 5. *$(A, \mathcal{I})$, where A is the set of atoms of a semi-modular lattice with initial element, is a matroid.*

Going back to the usual example, the vectors of a vector space are the atoms of the lattice of its subspaces. The height of a vector subspace (element of $L(V)$) is its dimension as a vector space. In this case (9) means just that a set of vectors $\{v_1, ..., v_n\}$ are $\mathcal{I}$ if they generate a space of dimension n: in other words, they are linearly independent!

According to Theorem 3, all collections of atoms are $\mathcal{I}$ in the lattice $L(\Theta)$. More interesting is the case of the lattice $L^*(\Theta)$ in which the order relation is the inverse of (7): $\Theta_1 \le \Theta_2$ iff Θ_1 is a coarsening of Θ_2. Reversing the ordering has dramatic effects on the properties of a lattice: in particular, on a semi-modular lattice it erases semi-modularity (Definition 4). More important to us, it alters the collection of sets which are independent according to (9), as both the function h and the sup $\vee$ change.

7.1 Anti-matroid of independent binary frames

Consider the partition lattice associated with a frame $\Theta = \{1, 2, 3, 4\}$ of cardinality 4. There cannot be $\mathcal{IF}$ collections of three or more frames, as in that case the size of their minimal refinement should be (by Equation 5) $2 \times 2 \times 2 = 8$ for them to be $\mathcal{IF}$. Let us then focus on all pairs of binary partitions (the atoms of $L^*(\Theta)$). By looking at Figure 5 we

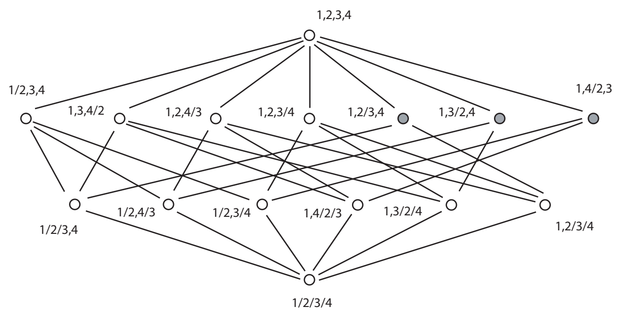

Figure 5: The partition lattice for a frame Θ of size 4. Partitions $A_1, ..., A_k$ of Θ are denoted by $A_1/.../A_k$. Partitions with the same number of elements are arranged on the same level. An edge between two nodes indicates that the bottom partition "covers" the top one (in $L^*(\Theta)$).

can notice that the only $\mathcal{IF}$ pairs are formed by two of the following binary partitions

$$1,2/3,4 \quad 1,3/2,4 \quad 1,4/2,3.$$

(dark nodes in the diagram) as their maximal coarsening is $\Theta = \{1, 2, 3, 4\}$ and has cardinality $2 \times 2 = 4$.

But it is easy to see that all other binary pairs are $\mathcal{I}$.

Theorem 4. *Pairs of binary partitions of a frame Θ (atoms of the lattice $L^*(\Theta)$) are independent as frames ($\mathcal{IF}$) if and only if they are not independent as elements of a matroid ($\mathcal{I}$).*

Proof. As in $L^*(\Theta)$ the sup is the minimal refinement (Cuzzolin 2007), (9) reads as

$$\begin{array}{c} h(\Theta_1 \otimes \cdots \otimes \Theta_n) = \sum_i h(\Theta_i) \equiv \\ |\Theta_1 \otimes \cdots \otimes \Theta_n| - 1 = \sum_i (|\Theta_i| - 1) \end{array} \qquad (10)$$

which for $n = 2$ reads as $|\Theta_1 \otimes \Theta_2| = |\Theta_1| + |\Theta_2| - 1$. For pairs of binary frames $\mathcal{IF}$ can be written as $|\Theta_1 \otimes \Theta_2| = |\Theta_1| \cdot |\Theta_2| = 2 \cdot 2 = 4$. For all the binary pairs which are not $\mathcal{IF}$ we have $|\Theta_1 \otimes \Theta_2| = 3$ (see Figure 5 again) and the equality is met. □

7.2 Mutual exclusivity of $\mathcal{I}$ and $\mathcal{IF}$

We can in fact prove a stronger, more general statement.

Theorem 5. *A collection of $\mathcal{IF}$ compatible frames $\Theta_1, ..., \Theta_n$ is $\mathcal{I}$ iff $n = 2$ and one of the frames is the trivial partition.*

Proof. According to Equation (5), $\Theta_1, ..., \Theta_n$ are $\mathcal{IF}$ iff $|\otimes \Theta_i| = \prod_i |\Theta_i|$, while according to (10) they are $\mathcal{I}$ iff $|\Theta_1 \otimes \cdots \otimes \Theta_n| - 1 = \sum_i (|\Theta_i| - 1)$. They are met together iff

$$\sum_i |\Theta_i| - \prod_i |\Theta_i| = n - 1$$

which happens only if $n = 2$ and one of Θ_1, Θ_2 has cardinality 1. □

7.3 A general feature?

Instead of being algebraically related notions, independence of frames and matroidicity work against each other. In a sense, we can say that collections of independent frames are "anti-matroids". As independence of frames derives from independence of Boolean subalgebras of a Boolean algebra (Sikorski 1964), this is likely to have interesting wider implications on the relationship between independence in those two fields of mathematics.

However, it is worth to notice that this analysis is valid for the specific matroid induced on the atoms by the independence relation (9). Although the latter is the classical structure usually associated with semi-modular lattices, we cannot rule out the existence of other matroidal structures on $L(\Theta)$, for which a direct extension of the above results would be premature.

8 Conclusions

In this paper we studied Shafer's notion of independence of frames (as an expression of independence of sources in evidence combination) in an algebraic setup. It turns out that $\mathcal{IF}$ cannot be explained neither in terms of classical matroidal independence, nor as a cryptomorphic form of independence of flats on geometric lattices. It turns in fact out that independence of frames is actually opposed to matroidal independence, a rather surprising result. The prosecution of this study could in the future shed some more light on both the nature of independence of sources in the theory of subjective probability, and the relationship between matroidal and Boolean independence in discrete mathematics, pointing out the necessity of a more general, comprehensive definition of this very important notion.

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