Three alternative combinatorial formulations of the theory of evidence

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Abstract

In this paper we introduce three alternative combinatorial formulations of the theory of evidence (ToE), by proving that both plausibility and commonality functions share the structure of "sum function" with belief functions. We compute their Moebius inverses, which we call basic plausibility and commonality assignments. In the framework of the geometric approach to uncertainty measures the equivalence of the associated formulations of the ToE is mirrored by the geometric congruence of the related simplices. We can then describe the point-wise geometry of these sum functions in terms of rigid transformations mapping them onto each other. Combination rules can be applied to plausibility and commonality functions through their Moebius inverses, leading to interesting applications of such inverses to the probabilistic transformation problem.

Keywords: theory of evidence; combinatorics; sum function; Moebius inverse; basic plausibility and commonality assignment; convex geometry; simplex; congruence.

1 Introduction

Uncertainty measures are of paramount importance in the field of artificial intelligence, where problems involving formalized reasoning are common. During the last decades a number of different descriptions of uncertainty have been proposed, as either alternatives to or extensions of classical probability theory. These include probability intervals [19], credal sets, monotone capacities [41], random sets [26]. New original foundations of subjective probability in behavioral terms [40, 39] or by means of game theory [31] have been brought forward. The *theory of evidence* (ToE) is one of the most popular such formalisms. It has been introduced in the late Seventies by Glenn Shafer [28] as a way of representing epistemic knowledge, starting from a sequence of seminal works [10, 12, 13] by Arthur Dempster. In this formalism uncertain states of knowledge are described by *belief functions* rather than probability distributions. Belief functions can be interpreted as collections of lower bounds to the probability values of an unknown probability distribution on all the subsets of a finite domain or "frame". Variants or continuous extensions of the ToE in terms of hints [23] or allocations of probability [29] have later been proposed.

From a combinatorial point of view, belief functions can be seen as *sum functions*, i.e., functions on the power set $2^\Theta = \{A \subseteq \Theta\}$ of a finite domain Θ

$$b(A) = \sum_{B \subseteq A} m_b(B)$$

induced by a non-negative *basic probability assignment* $m_b : 2^\Theta \to [0,1]$. The degree of belief of any event A is obtained by summing all the basic probabilities of its subsets B. Combinatorially, a basic probability assignment is the *Moebius inverse* of the related belief function [1].
From the point of view of subjective probability, the same evidence associated with a belief function is carried by the related plausibility

$$pl_b(A) = 1 - b(A^c)$$

and commonality

$$Q_b(A) = \sum_{B \supseteq A} m_b(B)$$

functions. Such equivalent (from an evidential perspective) functions, however, lack a similar coherent mathematical characterization.

1.1 Contribution

The contribution of this paper is twofold.
In the first part we introduce two alternative combinatorial formulations of the theory of evidence, by proving that both plausibility and commonality functions share with belief functions the structure of sum function. In particular we compute their Moebius inverses, which it is natural to call *basic plausibility* and *commonality assignments*, respectively.
Besides giving the overall mathematical structure of the theory of evidence an elegant symmetry, the notions of basic plausibility and commonality assignments turn out to be useful in problems involving the combination of plausibility or commonality functions. This is the case for the problem of transforming a belief function into a probability distribution [25, 32, 38, 37, 2, 3, 5], or when computing the canonical decomposition of support functions [35, 24].

In the second part we extend the geometric approach to the theory of evidence [7] to the introduced alternative combinatorial models. Indeed, belief functions can be seen as points of a simplex, in which their simplicial coordinates correspond to the associate Moebius inverse, the basic probability assignment. While such geometric analysis was so far restricted to belief measures, we show here that plausibility and commonality functions inherit the same simplicial geometry of belief functions, with their Moebius inverses playing again the role of simplicial coordinates. The equivalence of the associated formulations of the ToE is geometrically mirrored by the congruence of their simplices. In particular, the relation between upper and lower probabilities (so important in subjective probability) can be geometrically expressed as a simple rigid transformation.

1.2 Paper outline

First (Section 2) we recall the basic notions of the theory of evidence, in particular the key ideas of belief, plausibility, and commonality functions. In Section 3, the core of the paper, we introduce the basic plausibility (Section 3.1) and commonality (3.2) assignments as Moebius inverses of plausibility and commonality functions, respectively.

In Section 4, after briefly recalling the interpretation of belief functions as points of a Cartesian space, we show that the geometric approach to uncertainty can be extended to plausibility and commonality functions, and the simplicial structure of the related spaces recovered as a function of their Moebius inverses. We show (Section 5) that the equivalence of the proposed alternative formulations of the ToE is reflected by the congruence of the corresponding simplices in the geometric framework. We also discuss the point-wise geometry of the triplet (b, pl_b, Q_b) in terms of the rigid transformation mapping them onto each other, as a geometric nexus between the proposed models.

Finally, in Section 6 some interesting applications of the notion of basic plausibility assignment to the probabilistic approximation problem are analyzed in some detail. We consider in particular the cases of the relative belief of singletons [6] and the intersection probability [5].

2 Belief, plausibility, and commonality functions

Even though belief functions can be given several alternative but equivalent definitions in terms of multi-valued mappings [30], random sets [26, 20], or inner measures [27, 15], in Shafer's formulation [28] a central role is played by the notion of *basic probability assignment*.

A *basic probability assignment* (b.p.a.) over a finite set (*frame of discernment* [28]) Θ is a function $m : 2^\Theta \to [0, 1]$ on its power set $2^\Theta = \{A \subseteq \Theta\}$ such that

$$m(\emptyset) = 0, \quad \sum_{A \subseteq \Theta} m(A) = 1, \quad m(A) \geq 0 \ \forall A \subseteq \Theta.$$

Subsets of Θ associated with non-zero values of m are called *focal elements*.

The *belief function* (b.f.) $b : 2^\Theta \to [0, 1]$ associated with a basic probability assignment m_b on Θ is defined as:

$$b(A) = \sum_{B \subseteq A} m_b(B). \qquad (1)$$

A probability function (or *Bayesian* belief function) is a special b.f. assigning non-zero masses only to singletons : $m_b(A) = 0$, $|A| > 1$.

Functions of the form (1) on a partially ordered set are called *sum functions* [1]. A belief function b is therefore the sum function associated with a basic probability assignment m_b on the partially ordered set $(2^\Theta, \subseteq)$.
Conversely, the unique b.p.a. m_b associated with a given belief function b can be recovered by means of the *Moebius inversion formula*

$$m_b(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} b(B). \tag{2}$$

Therefore, there is a 1-1 correspondence between the two set functions $m_b \leftrightarrow b$.

A dual mathematical representation of the evidence encoded by a belief function b is provided by the *plausibility function* (pl.f.) $pl_b : 2^\Theta \rightarrow [0,1]$, $A \mapsto pl_b(A)$, where the plausibility $pl_b(A)$ of an event A is given by

$$pl_b(A) \doteq 1 - b(A^c) = 1 - \sum_{B \subseteq A^c} m_b(B) = \sum_{B \cap A \neq \emptyset} m_b(B) \geq b(A) \tag{3}$$

and expresses the amount of evidence *not against* A. We will denote by m_b, pl_b the b.p.a. and pl.f. uniquely associated with a belief function b.

A third mathematical model of the evidence carried by a belief function is represented by the *commonality function* (comm.f.) $Q_b : 2^\Theta \rightarrow [0,1]$, $A \mapsto Q_b(A)$, where the *commonality number* $Q_b(A)$ can be interpreted as the amount of mass which can move freely through the entire event A,

$$Q_b(A) \doteq \sum_{B \supseteq A} m_b(B). \tag{4}$$

2.1 Example

Let us consider a belief function b on a frame of size 3, $\Theta = \{x, y, z\}$ with b.p.a. (see Figure 1) $m_b(x) = 1/3$, $m_b(\Theta) = 2/3$.

The belief values of b on all the possible events of Θ are, according to Equation (1),

$$\begin{array}{lll} b(\Theta) = m_b(x) + m_b(\Theta) = 1, & b(x) = m_b(x) = 1/3, & b(y) = 0,\ b(z) = 0, \\ b(\{x,y\}) = m_b(x) = 1/3, & b(\{x,z\}) = m_b(x) = 1/3, & b(\{y,z\}) = 0. \end{array}$$

To appreciate the difference between belief, plausibility, and commonality values let us consider in particular the event $A = \{x, y\}$. Its belief value

$$b(\{x,y\}) = \sum_{B \subseteq \{x,y\}} m_b(B) = m_b(x) = 1/3$$

represents the amount of evidence which *surely supports* $\{x, y\}$. The total mass $b(A)$ is guaranteed to involve *only* elements of $A = \{x, y\}$. On the other hand,

$$pl_b(\{x,y\}) = 1 - b(\{x,y\}^c) = \sum_{B \cap \{x,y\} \neq \emptyset} m_b(B) = 1 - b(z) = 1$$

measures the evidence *not surely against* it. The plausibility value $pl_b(A)$ accounts for the mass that *might* be assigned to some element of A.
Finally, the commonality number

$$Q_b(\{x,y\}) = \sum_{A \supseteq \{x,y\}} m_b(A) = m_b(\Theta) = 2/3$$

indicates the amount of evidence which can (possibly) *equally support* each of the outcomes in $A = \{x, y\}$ (i.e., x and y).

3 Two alternative formulations of the theory of evidence

Plausibility and commonality functions are both equivalent representations of the evidence carried by a belief function. It is therefore natural to guess that they should share with belief functions the combinatorial form of sum function on the power set 2^Θ. Belief functions encode evidence by cumulating basic probabilities on intervals of events $\{B \subseteq A\}$, yielding a collection of belief values $b(A) = \sum_{B \subseteq A} m(B)$. We show here that we can indeed represent the same evidence in terms of a basic plausibility (commonality) assignment on the power set, and compute the related plausibility (commonality) set function by integrating the basic assignment over similar intervals.

3.1 Basic plausibility assignment

Let us define the Moebius inverse $\mu_b : 2^\Theta \rightarrow \mathbb{R}$ of a plausibility function pl_b:

$$\mu_b(A) \doteq \sum_{B \subseteq A} (-1)^{|A \setminus B|} pl_b(B) \tag{5}$$

so that

$$pl_b(A) = \sum_{B \subseteq A} \mu_b(B). \tag{6}$$

It is natural to call the function $\mu_b : 2^\Theta \rightarrow \mathbb{R}$ defined by expression (5) *basic plausibility assignment* (b.pl.a.). PL.F.s are then sum functions on 2^Θ of the form (6), whose Moebius inverse is the b.pl.a. (5). Basic probabilities and plausibilities are obviously related.

Theorem 1.

$$\mu_b(A) = \begin{cases} (-1)^{|A|+1} \sum_{C \supseteq A} m_b(C) & A \neq \emptyset \\ 0 & A = \emptyset. \end{cases} \tag{7}$$

Proof. The definition (3) of plausibility function yields

$$\mu_b(A) = \sum_{B\subseteq A}(-1)^{|A\setminus B|}pl_b(B) = \sum_{B\subseteq A}(-1)^{|A\setminus B|}(1-b(B^c)) = \sum_{B\subseteq A}(-1)^{|A\setminus B|} + \\ -\sum_{B\subseteq A}(-1)^{|A\setminus B|}b(B^c) = 0 - \sum_{B\subseteq A}(-1)^{|A\setminus B|}b(B^c) = -\sum_{B\subseteq A}(-1)^{|A\setminus B|}b(B^c)$$

since by Newton's binomial theorem $\sum_{B\subseteq A}(-1)^{|A\setminus B|} = 0$ if $A \neq \emptyset$, $(-1)^{|A|}$ otherwise. If $B \subseteq A$ then $B^c \supseteq A^c$, so that the above expression becomes

$$-\sum_{\emptyset\subsetneq B\subseteq A}(-1)^{|A\setminus B|}\Big(\sum_{C\subseteq B^c} m_b(C)\Big) = -\sum_{C\subseteq\Theta} m_b(C)\Big(\sum_{B:B\subseteq A, B^c\supseteq C}(-1)^{|A\setminus B|}\Big) = \\ = -\sum_{C\subseteq\Theta} m_b(C)\Big(\sum_{B\subseteq A\cap C^c}(-1)^{|A\setminus B|}\Big) \tag{8}$$

for $B^c \supseteq C, B \subseteq A$ is equivalent to $B \subseteq C^c, B \subseteq A \equiv B \subseteq (A\cap C^c)$.
Let us analyze the following function of C:

$$f(C) \doteq \sum_{B\subseteq A\cap C^c}(-1)^{|A\setminus B|}.$$

If $A\cap C^c = \emptyset$ then $B = \emptyset$ and the sum is equal to $f(C) = (-1)^{|A|}$. If $A\cap C^c \neq \emptyset$, instead, we can write $D \doteq C^c \cap A$ and obtain

$$f(C) = \sum_{B\subseteq D}(-1)^{|A\setminus B|} = \sum_{B\subseteq D}(-1)^{|A\setminus D|+|D\setminus B|},$$

since $B \subseteq D \subseteq A$ and $|A| - |B| = |A| - |D| + |D| - |B|$. But then

$$f(C) = (-1)^{|A|-|D|}\sum_{B\subseteq D}(-1)^{|D|-|B|} = 0,$$

given that $\sum_{B\subseteq D}(-1)^{|D|-|B|} = 0$ by Newton's binomial formula again.
In summary, $f(C) = 0$ if $C^c \cap A \neq \emptyset$, $f(C) = (-1)^{|A|}$ if $C^c \cap A = \emptyset$. We can then rewrite (8) as:

$$-\sum_{C\subseteq\Theta} m_b(C)f(C) = -\sum_{C:C^c\cap A\neq\emptyset} m_b(C)\cdot 0 - \sum_{C:C^c\cap A=\emptyset} m_b(C)\cdot(-1)^{|A|} = \\ = (-1)^{|A|+1}\sum_{C:C^c\cap A=\emptyset} m_b(C) = (-1)^{|A|+1}\sum_{C\supseteq A} m_b(C).$$

□

As b.p.a.s do, basic plausibility assignments meet the normalization constraint. In other words, pl.f.s are *normalized sum functions* [1]:

$$\sum_{A\subseteq\Theta}\mu_b(A) = -\sum_{\emptyset\subsetneq A\subseteq\Theta}(-1)^{|A|}\sum_{C\supseteq A} m_b(C) = -\sum_{C\subseteq\Theta} m_b(C)\cdot\sum_{\emptyset\subsetneq A\subseteq C}(-1)^{|A|} = 1$$

since

$$-\sum_{\emptyset\subsetneq A\subseteq C}(-1)^{|A|}=-(0-(-1)^0)=1$$

again by Newton's binomial theorem:

$$\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}=(p+q)^n. \tag{9}$$

However, unlike its counterpart m_b, μ_b is not guaranteed to be non-negative.

3.1.1 Example of basic plausibility assignment

Let us consider as an example a belief function b on a binary frame $\Theta_2=\{x,y\}$ with b.p.a.:

$$m_b(x)=\frac{1}{3},\ \ m_b(\Theta)=\frac{2}{3}.$$

Using Equation (5) we can compute its basic plausibility assignment as follows:

$$\begin{array}{lllll}
\mu_b(x) & = & (-1)^{|x|+1}\sum_{C\supseteq\{x\}}m_b(C) & = (-1)^2(m_b(x)+m_b(\Theta))=1,\\
\mu_b(y) & = & (-1)^{|y|+1}\sum_{C\supseteq\{y\}}m_b(C) & = (-1)^2 m_b(\Theta)=2/3,\\
\mu_b(\Theta) & = & (-1)^{|\Theta|+1}\sum_{C\supseteq\Theta}m_b(C) & = (-1)m_b(\Theta)=-2/3<0.
\end{array}$$

This confirms that b.pl.a.s meet the normalization constraint but not the non-negativity one.

3.1.2 Relation between basic probability and plausibility assignments

Basic probability and plausibility assignments are linked by a rather elegant relation.

Theorem 2.

$$\sum_{A\supseteq\{x\}}\mu_b(A)=m_b(x). \tag{10}$$

Proof.

$$\begin{array}{rcl}
\sum_{A\supseteq\{x\}}\mu_b(A) & = & \sum_{A\supseteq\{x\}}(-1)^{|A|+1}\Big(\sum_{B\supseteq A}m_b(B)\Big)\\
 & = & -\sum_{B\supseteq\{x\}}m_b(B)\Big(\sum_{\{x\}\subseteq A\subseteq B}(-1)^{|A|}\Big)
\end{array}$$

where, by Newton's binomial $\sum_{k=0}^{n}1^{n-k}(-1)^k=0$,

$$\sum_{\{x\}\subseteq A\subseteq B}(-1)^{|A|}=\left\{\begin{array}{ll}0 & B\neq\{x\}\\ -1 & B=\{x\}.\end{array}\right.$$

□

3.2 Basic commonality assignment

It is straightforward to prove that commonality functions are also sum functions and sport some interesting similarities with plausibility functions. Let us define the Moebius inverse $q_b : 2^\Theta \rightarrow \mathbb{R}$, $B \mapsto q_b(B)$ of a commonality function Q_b as:

$$q_b(B) = \sum_{\emptyset \subseteq A \subseteq B} (-1)^{|B \setminus A|} Q_b(A). \tag{11}$$

It is natural to call the quantity (11) the *basic commonality assignment* (or b.comm.a.) associated with a belief function b. To arrive at its explicit form we just need to replace the definition of $Q_b(A)$ into (11). We obtain:

$$\begin{aligned} q_b(B) &= \sum_{\emptyset \subseteq A \subseteq B} (-1)^{|B \setminus A|} \Bigg(\sum_{C \supseteq A} m_b(C) \Bigg) \\ &= \sum_{\emptyset \subsetneq A \subseteq B} (-1)^{|B \setminus A|} \Bigg(\sum_{C \supseteq A} m_b(C) \Bigg) + (-1)^{|B|-|\emptyset|} \sum_{C \supseteq \emptyset} m_b(C) \\ &= \sum_{B \cap C \neq \emptyset} m_b(C) \Bigg(\sum_{\emptyset \subsetneq A \subseteq B \cap C} (-1)^{|B \setminus A|} \Bigg) + (-1)^{|B|}. \end{aligned}$$

But now, since $B \setminus A = B \setminus C + B \cap C \setminus A$, we have that:

$$\begin{aligned} \sum_{\emptyset \subsetneq A \subseteq (B \cap C)} (-1)^{|B \setminus A|} &= (-1)^{|B \setminus C|} \sum_{\emptyset \subsetneq A \subseteq (B \cap C)} (-1)^{|B \cap C| - |A|} \\ &= (-1)^{|B \setminus C|} \Big[(1-1)^{|B \cap C|} - (-1)^{|B \cap C| - |\emptyset|} \Big] = \end{aligned}$$

$= (-1)^{|B|+1}$. Therefore, the b.comm.a. $q_b(B)$ can be expressed as:

$$\begin{aligned} q_b(B) &= (-1)^{|B|+1} \sum_{B \cap C \neq \emptyset} m_b(C) + (-1)^{|B|} = (-1)^{|B|} \Bigg(1 - \sum_{B \cap C \neq \emptyset} m_b(C) \Bigg) \\ &= (-1)^{|B|} (1 - pl_b(B)) = (-1)^{|B|} b(B^c) \end{aligned} \tag{12}$$

(note that $q_b(\emptyset) = (-1)^{|\emptyset|} b(\emptyset) = 1$).

3.2.1 Properties of basic commonality assignments

Basic commonality assignments *do not* meet the normalization axiom, as

$$\sum_{\emptyset \subseteq B \subseteq \Theta} q_b(B) = Q_b(\Theta) = m_b(\Theta).$$

In other words, whereas belief functions are normalized sum functions (n.s.f.) with non-negative Moebius inverse, and plausibility functions are normalized sum functions, commonality functions are combinatorially *unnormalized* sum functions. Going back to the example of Section 3.1.1, the b.comm.a. associated with $m_b(x) = 1/3$, $m_b(\Theta) = 2/3$ is (by Equation (12))

$$\begin{array}{ll} q_b(\emptyset) = (-1)^{|\emptyset|} b(\Theta) = 1, & q_b(x) = (-1)^{|x|} b(y) = -m_b(y) = 0, \\ q_b(\Theta) = (-1)^{|\Theta|} b(\emptyset) = 0, & q_b(y) = (-1)^{|y|} b(x) = -m_b(x) = -1/3 \end{array}$$

so that

$$\sum_{\emptyset \subseteq B \subseteq \Theta} q_b(B) = 1 - 1/3 = 2/3 = m_b(\Theta) = Q_b(\Theta).$$

4 Geometry of plausibilities and commonalities

The theory of evidence can be given alternative formulations in terms of plausibility and commonality assignments. As a consequence, plausibility and commonality functions (just like belief functions [7, 5, 8]) can be given simple but elegant geometric descriptions in terms of generalized triangles or *simplices*.

4.1 Belief and plausibility functions as vectors

A belief function $b : 2^\Theta \rightarrow [0, 1]$ on a frame of discernment Θ is completely specified by its $N - 2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$, $N \doteq 2^{|\Theta|}$ (as $b(\emptyset) = 0$, $b(\Theta) = 1$ for all b). We can collect all such belief values in a vector of $\mathbb{R}^{N-2}$. Similarly, plausibility functions are completely specified by their $N - 2$ plausibility values $\{pl_b(A), \emptyset \subsetneq A \subsetneq \Theta\}$ and can also be represented as vectors of $\mathbb{R}^{N-2}$. We can therefore associate a pair of belief b and plausibility pl_b functions with the following vectors, which we still denote by b and pl_b:

$$b = \sum_{\emptyset \subsetneq A \subseteq \Theta} b(A) X_A, \quad pl_b = \sum_{\emptyset \subsetneq A \subsetneq \Theta} pl_b(A) X_A, \tag{13}$$

where $\{X_A : \emptyset \subsetneq A \subsetneq \Theta\}$ is a reference frame in the Cartesian space $\mathbb{R}^{N-2}$.

It can be proven that, in the case of belief functions, each vector b can be written as a convex combination of the vectors b_A representing all *categorical* [36] belief functions b_A [7]

$$b = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) b_A, \tag{14}$$

where $m_{b_A}(A) = 1$, $m_{b_A}(B) = 0$ for all $B \neq A$. Let us call *belief space* the set $\mathcal{B}$ of points of $\mathbb{R}^{N-2}$ which do correspond to a b.f. It can be proven that [7]:

Proposition 1. *The belief space $\mathcal{B}$ is a convex set of the form*

$$\mathcal{B} = Cl(b_A, \ \emptyset \subsetneq A \subseteq \Theta) \tag{15}$$

where Cl denotes the convex closure operator:

$$Cl(b_1, ..., b_k) = \Big\{ b \in \mathcal{B} : b = \alpha_1 b_1 + \cdots + \alpha_k b_k, \sum_i \alpha_i = 1, \ \alpha_i \geq 0 \ \forall i \Big\}.$$

In convex geometry, a k-dimensional *simplex* $Cl(x_1, ..., x_{k+1})$ is the convex closure of $k+1$ affinely independent[1] points $x_1, ..., x_{k+1}$ of the Cartesian space $\mathbb{R}^k$. It is easy to see that the vectors $\{b_A, \emptyset \subsetneq A \subseteq \Theta\}$ representing all the categorical belief functions are affinely independent in $\mathbb{R}^{N-2}$. It follows that $\mathcal{B}$ is a simplex there. The region $\mathcal{P}$ of all probability measures on the same domain Θ is a face of $\mathcal{B}$, namely: $\mathcal{P} = Cl(b_x, x \in \Theta)$.

It can be noticed that the simplicial coordinates of b in the categorical reference frame are given by m_b, i.e., the Moebius inverse of b. The simplicial form of $\mathcal{B}$ is the geometric counterpart of the nature of belief functions as sum functions admitting Moebius inverse.

4.1.1 The case of unnormalized belief functions

In the practical use of the theory of evidence, people often consider *unnormalized belief functions* (u.b.f.s) [33], i.e., belief functions $b(A) = \sum_{\emptyset \neq B \subseteq A} m_b(B)$ admitting non-zero support $m_b(\emptyset) \neq 0$ for the empty set $\emptyset$. The latter is an indicator of the amount of internal conflict of the evidence carried by a belief function, or of the possibility that the current frame of discernment does not exhaust all the possible outcomes of the problem.
Unnormalized belief functions are naturally associated with vectors with $N = 2^{|\Theta|}$ coordinates, as $b(\emptyset)$ cannot be neglected anymore. We can then extend the set of categorical belief functions as follows

$$\{b_A \in \mathbb{R}^N, \emptyset \subseteq A \subseteq \Theta\},$$

this time including a new vector $b_\emptyset \doteq [1\ 0\ \cdots 0]'$. Note also that in this case $b_\Theta = [0 \cdots 0\ 1]'$. The space of unnormalized b.f.s is again a simplex in $\mathbb{R}^N$, namely $\mathcal{B}^U = Cl(b_A, \emptyset \subseteq A \subseteq \Theta)$.

4.2 The geometry of plausibility functions

4.2.1 Plausibility assignment and simplicial coordinates

The categorical belief functions $\{b_A : \emptyset \subsetneq A \subsetneq \Theta\}$ form a set of independent vectors in $\mathbb{R}^{N-2}$, so that the collections $\{X_A\}$ and $\{b_A\}$ represent two distinct coordinate frames in that Cartesian space. To understand where a plausibility vector is located in the categorical reference frame $\{b_A, \emptyset \subsetneq A \subsetneq \Theta\}$ we need to compute the coordinate change between such frames.

Lemma 1. *The coordinate change between the two coordinate frames* $\{X_A : \emptyset \subsetneq A \subsetneq \Theta\}$ *and* $\{b_A : \emptyset \subsetneq A \subsetneq \Theta\}$ *is given by*

$$X_A = \sum_{B \supseteq A} (-1)^{|B \setminus A|} b_B. \qquad (16)$$

[1] An *affine combination* of k points $v_1, \cdots, v_k \in \mathbb{R}^m$ is a sum $\alpha_1 v_1 + \cdots + \alpha_k v_k$ whose coefficients sum to one: $\sum_i \alpha_i = 1$. The affine subspace generated by the points $v_1, \cdots, v_k \in \mathbb{R}^m$ is the set

$$\Big\{ v \in \mathbb{R}^m : v = \alpha_1 v_1 + \cdots + \alpha_k v_k, \sum_i \alpha_i = 1 \Big\}.$$

If $v_1, ..., v_k$ generate an affine space of dimension k they are said to be *affinely independent*.

We can use Lemma 1 to find the coordinates of a plausibility function in the categorical reference frame, by putting the corresponding vector pl_b (13) in the form of Equation (14).
By replacing expression (16) for X_A into Equation (13) we get

$$\begin{array}{rcl} pl_b & = & \displaystyle\sum_{\emptyset\subsetneq A\subsetneq\Theta} pl_b(A)X_A = \sum_{\emptyset\subsetneq A\subsetneq\Theta} pl_b(A)\Big(\sum_{B\supseteq A} b_B(-1)^{|B\setminus A|}\Big) \\ & = & \displaystyle\sum_{\emptyset\subsetneq B\subsetneq\Theta} b_B\Big(\sum_{A\subseteq B}(-1)^{|B-A|}pl_b(A)\Big) = \sum_{\emptyset\subsetneq A\subsetneq\Theta}\mu_b(A)b_A, \end{array} \tag{17}$$

where we used the definition (5) of basic plausibility assignment and we inverted the role of A and B for sake of homogeneity of the notation. Incidentally, as $b_\Theta = [0, \cdots, 0]' = \mathbf{0}$ is the origin of $\mathbb{R}^{N-2}$, we can also write:

$$pl_b = \sum_{\emptyset\subsetneq A\subseteq\Theta}\mu_b(A)b_A$$

(including Θ). Just like in Equation (14), it can be noticed that the coordinates of pl_b in the categorical frame are given by its Moebius inverse, the basic plausibility assignment.

4.2.2 Plausibility space

Let us call *plausibility space* the region $\mathcal{PL}$ of $\mathbb{R}^{N-2}$ whose points correspond to admissible pl.f.s.

Theorem 3. *The plausibility space $\mathcal{PL}$ is a simplex $\mathcal{PL} = Cl(pl_A, \emptyset \subsetneq A \subseteq \Theta)$ whose vertices can be expressed in terms of the categorical belief functions (the vertices of the belief space) as*

$$pl_A = -\sum_{\emptyset\subsetneq B\subseteq A}(-1)^{|B|}b_B. \tag{18}$$

Proof. We just need to rewrite expression (17) as a convex combination of points. We get (by Equation (7)):

$$\begin{array}{rcl} pl_b & = & \displaystyle\sum_{\emptyset\subsetneq A\subseteq\Theta}\mu_b(A)b_A = \sum_{\emptyset\subsetneq A\subseteq\Theta}(-1)^{|A|+1}\Big(\sum_{C\supseteq A} m_b(C)\Big)b_A \\ & = & \displaystyle\sum_{\emptyset\subsetneq A\subseteq\Theta}(-1)^{|A|+1}b_A\Big(\sum_{C\supseteq A} m_b(C)\Big) \\ & = & \displaystyle\sum_{\emptyset\subsetneq C\subseteq\Theta} m_b(C)\Big(\sum_{\emptyset\subsetneq A\subseteq C}(-1)^{|A|+1}b_A\Big) = \sum_{\emptyset\subsetneq C\subseteq\Theta} m_b(C)pl_C. \end{array} \tag{19}$$

The latter is indeed a convex combination, since basic probability assignments are non-negative (but $m_b(\emptyset = 0)$) and have unitary sum. Accordingly,

$$\begin{array}{rcl} \mathcal{PL} & = & \{pl_b, b\in\mathcal{B}\} \\ & = & \Big\{\displaystyle\sum_{\emptyset\subsetneq C\subseteq\Theta} m_b(C)pl_C,\ \sum_C m_b(C) = 1,\ m_b(C)\geq 0\ \forall C\subseteq\Theta\Big\} \\ & = & Cl(pl_A, \emptyset\subsetneq A\subseteq\Theta) \end{array}$$

(after exchanging C with A to keep the notation consistent). $\square$

It is easy to note that $pl_x = -(-1)^{|x|} b_x = b_x \ \forall x \in \Theta$, so that $\mathcal{B} \cap \mathcal{PL} \supset \mathcal{P}$. The vertices of the plausibility space have a natural interpretation.

Theorem 4. *The vertex pl_A of the plausibility space is the plausibility vector associated with the categorical belief function b_A: $pl_A = pl_{b_A}$.*

Proof. Expression (18) is equivalent to

$$pl_A(C) = -\sum_{\emptyset \subsetneq B \subseteq A} (-1)^{|B|} b_B(C)$$

$\forall C \subseteq \Theta$. But since $b_B(C) = 1$ if $C \supseteq B$, 0 otherwise, we have that

$$pl_A(C) = -\sum_{B \subseteq A, B \subseteq C, B \neq \emptyset} (-1)^{|B|} = -\sum_{\emptyset \subsetneq B \subseteq A \cap C} (-1)^{|B|}.$$

Now, if $A \cap C = \emptyset$ then there are no addenda in the above sum, which goes to zero. Otherwise, by Newton's binomial formula (9), we have:

$$pl_A(C) = -\{[1 + (-1)]^{|A \cap C|} - (-1)^0\} = 1.$$

On the other side, by definition of plausibility function,

$$pl_{b_A}(C) = \sum_{B \cap C \neq \emptyset} m_{b_A}(B) = \begin{cases} 1 & A \cap C \neq \emptyset \\ 0 & A \cap C = \emptyset \end{cases}$$

and the two quantities coincide. $\square$

4.2.3 Unnormalized case

It is interesting to consider also the case of unnormalized belief functions.
As a matter of fact, it can be seen that Theorems 1 and 4 fully retain their validity. However, in the case of Theorem 3, as in general $m_b(\emptyset) \neq 0$ we need to modify Equation (19) by adding a term related to the empty set. This yields

$$pl_b = \sum_{\emptyset \subsetneq C \subseteq \Theta} m_b(C) pl_C + m_b(\emptyset) pl_\emptyset$$

where pl_C, $C \neq \emptyset$ is still given by Equation (18), and $pl_\emptyset = \mathbf{0}$ is the origin of $\mathbb{R}^N$. Note that even in the case of unnormalized belief functions (Equation (18)) the empty set is not considered, for $\mu(\emptyset) = 0$.

4.2.4 Toy example: the binary case

Figure 2 shows the geometry of belief and plausibility spaces in the simple case study of a binary frame $\Theta_2 = \{x, y\}$, where belief and plausibility vectors are points of a plane $\mathbb{R}^2$ with coordinates

$$\begin{array}{rcl} b & = & [b(x) = m_b(x), b(y) = m_b(y)]' \\ pl_b & = & [pl_b(x) = 1 - m_b(y), pl_b(y) = 1 - m_b(x)]', \end{array}$$

respectively. They form two simplices (triangles)

$$\begin{array}{rcl}\mathcal{B} & = & Cl(b_\Theta = [0,0]' = \mathbf{0}, b_x, b_y), \\ \mathcal{PL} & = & Cl(pl_\Theta = [1,1]' = \mathbf{1}, pl_x = b_x, pl_y = b_y)\end{array}$$

which are symmetric with respect to the probability simplex $\mathcal{P}$ (in this case a segment) and congruent, so that they can be moved onto each other by means of a rigid transformation. In this simple case such transformation is just a reflection through the Bayesian segment $\mathcal{P}$.
From Figure 2 it is clear that each pair of belief/plausibility functions (b, pl_b) determines a line $a(b, pl_b)$ which is orthogonal to $\mathcal{P}$, on which they lay on symmetric positions on the two sides of the Bayesian segment.

4.3 The geometry of commonality functions

In the case of commonality functions, as

$$Q_b(\emptyset) = \sum_{A \supseteq \emptyset} m_b(A) = \sum_{A \subseteq \Theta} m_b(A) = 1, \qquad Q_b(\Theta) = \sum_{A \supseteq \Theta} m_b(A) = m_b(\Theta)$$

each comm.f. Q_b needs $2^{|\Theta|} = N$ coordinates to be represented. The geometric counterpart of a commonality function is therefore the following vector of $\mathbb{R}^N$

$$Q_b = \sum_{\emptyset \subseteq A \subseteq \Theta} Q_b(A) X_A,$$

where $\{X_A : \emptyset \subseteq A \subseteq \Theta\}$ is the extended reference frame introduced in the case of unnormalized belief functions ($A = \Theta, \emptyset$ this time included).

As before we can use Lemma 1 to change the coordinate base and get the coordinates of Q_b with respect to the base $\{b_A, \emptyset \subseteq A \subseteq \Theta\}$ formed by all the categorical u.b.f.s. We get

$$\begin{array}{rcl}Q_b & = & \displaystyle\sum_{\emptyset \subseteq A \subseteq \Theta} Q_b(A) \Big(\sum_{B \supseteq A} b_B (-1)^{|B \setminus A|} \Big) \\ & = & \displaystyle\sum_{\emptyset \subseteq B \subseteq \Theta} b_B \Big(\sum_{A \subseteq B} (-1)^{|B \setminus A|} Q_b(A) \Big) = \sum_{\emptyset \subseteq B \subseteq \Theta} q_b(B) b_B\end{array}$$

where q_b is the basic commonality assignment (11).

Once again, we can use the explicit form (12) of a basic commonality assignment to recover the shape of the space $\mathcal{Q} \subset \mathbb{R}^N$ of all the commonality functions. We obtain:

$$\begin{array}{rcl}Q_b & = & \displaystyle\sum_{\emptyset \subseteq B \subseteq \Theta} (-1)^{|B|} b_B \Big(\sum_{\emptyset \subseteq A \subseteq B^c} m_b(A) \Big) \\ & = & \displaystyle\sum_{\emptyset \subseteq A \subseteq \Theta} m_b(A) \Big(\sum_{\emptyset \subseteq B \subseteq A^c} (-1)^{|B|} b_B \Big) = \sum_{\emptyset \subseteq A \subseteq \Theta} m_b(A) Q_A,\end{array}$$

where

$$Q_A \doteq \sum_{\emptyset \subseteq B \subseteq A^c} (-1)^{|B|} b_B \tag{20}$$

is the A-th vertex of the *commonality space*, which is hence given by

$$\mathcal{Q} = Cl(Q_A, \emptyset \subseteq A \subseteq \Theta).$$

Again, Q_A is the commonality function associated with the categorical belief function b_A, i.e.,

$$Q_{b_A} = \sum_{\emptyset \subseteq B \subseteq \Theta} q_{b_A}(B) b_B.$$

Indeed $q_{b_A}(B) = (-1)^{|B|}$ if $B^c \supseteq A$ (i.e., $B \subseteq A^c$), while $q_{b_A}(B) = 0$ otherwise, so that the two quantities coincide:

$$Q_{b_A} = \sum_{\emptyset \subseteq B \subseteq A^c} (-1)^{|B|} b_B = Q_A.$$

4.3.1 Running example: the binary case

In the binary case the commonality space $\mathcal{Q}_2$ needs $N = 2^2 = 4$ coordinates to be represented. Each commonality vector $Q_b = [Q_b(\emptyset), Q_b(x), Q_b(y), Q_b(\Theta)]'$ is such that:

$$\begin{array}{ll} Q_b(\emptyset) = 1, & Q_b(x) = \displaystyle\sum_{A \supseteq \{x\}} m_b(A) = pl_b(x), \\ Q_b(\Theta) = m_b(\Theta), & Q_b(y) = \displaystyle\sum_{A \supseteq \{y\}} m_b(A) = pl_b(y). \end{array}$$

The commonality space $\mathcal{Q}_2$ can then be drawn (if we neglect the coordinate $Q_b(\emptyset)$ which is constant $\forall b$) as in Figure 3. The vertices of $\mathcal{Q}_2$ are, according to Equation (20):

$$\begin{array}{rcl} Q_\emptyset & = & \displaystyle\sum_{\emptyset \subseteq B \subseteq \Theta} (-1)^{|B|} b_B = b_\emptyset + b_\Theta - b_x - b_y \\ & = & [1,1,1,1]' + [0,0,0,1]' - [0,1,0,1]' - [0,0,1,1]' = [1,0,0,0]' = Q_{b_\emptyset}, \\ Q_x & = & \displaystyle\sum_{\emptyset \subseteq B \subseteq \{y\}} (-1)^{|B|} b_B = b_\emptyset - b_y = [1,1,1,1]' - [0,0,1,1]' \\ & = & [1,1,0,0]' = Q_{b_x}, \\ Q_y & = & \displaystyle\sum_{\emptyset \subseteq B \subseteq \{x\}} (-1)^{|B|} b_B = b_\emptyset - b_x = [1,1,1,1]' - [0,1,0,1]' \\ & = & [1,0,1,0]' = Q_{b_y}. \end{array}$$

5 Equivalence and congruence

Both plausibility and commonality functions can then be thought of as sum functions on the partially ordered set 2^Θ (even though whereas belief and plausibility functions are normalized sum functions, comm.f.s are not). This in turn allows to describe them as points of some simplices $\mathcal{B}, \mathcal{PL}$ and $\mathcal{Q}$ in a Cartesian space. In fact, the *equivalence* of such alternative models of the ToE is geometrically mirrored by the *congruence* of the associated simplices.

5.1 Congruence

We have seen that in the case of a binary frame of discernment, $\mathcal{B}$ and $\mathcal{PL}$ are congruent, i.e. they can be superposed by means of a rigid transformation (see Section 4.2.4). The congruence of belief, plausibility and commonality spaces is indeed a general property.

5.1.1 Congruence of belief and plausibility spaces

Theorem 5. *The corresponding 1-dimensional faces $Cl(b_A, b_B)$ and $Cl(pl_A, pl_B)$ of belief and plausibility spaces are congruent, namely*

$$\|pl_B - pl_A\|_p = \|b_A - b_B\|_p$$

where $\| \|_p$ denotes the classical norm $\|\mathbf{v}\|_p \doteq \sqrt[p]{\sum_{i=1}^N |v_i|^p}$, $p = 1, 2, ..., +\infty$.

Proof. This a direct consequence of the definition of plausibility function. Let us denote by C, D two generic subsets of Θ. As $pl_A(C) = 1 - b_A(C^c)$ we have that $b_A(C^c) = 1 - pl_A(C)$, which in turn implies

$$b_A(C^c) - b_B(C^c) = 1 - pl_A(C) - 1 + pl_B(C) = pl_B(C) - pl_A(C).$$

Therefore, $\forall p$:

$$\sum_{C \subseteq \Theta} |pl_B(C) - pl_A(C)|^p = \sum_{C \subseteq \Theta} |b_A(C^c) - b_B(C^c)|^p = \sum_{D \subseteq \Theta} |b_A(D) - b_B(D)|^p.$$

□

Notice that the proof of Theorem 5 holds no matter if the pair $(\emptyset, \emptyset^c = \Theta)$ is considered or not. A straightforward consequence is that

Corollary 1. *$\mathcal{B}$ and $\mathcal{PL}$ are congruent; $\mathcal{B}^U$ and $\mathcal{PL}^U$ are congruent.*

as their corresponding 1-dimensional faces have the same length. This is due to the generalization of a well-known Euclid's theorem stating that triangles with sides of the same length are congruent. It is worth to notice that, although this holds for *simplices* (generalized triangles), the same is not true for *polytopes* in general, i.e. convex closures of a number of vertices greater than $n + 1$ where n is the dimension of the Cartesian space in which they are defined (think of a square and a rhombus both with sides of length 1).

5.1.2 Running example: the binary case

In the case of *unnormalized* belief functions belief, plausibility and commonality spaces all have $N = 2^{|\Theta|}$ vertices and dimension $N-1$:

$$\mathcal{B}^U = Cl(b_A, \emptyset \subseteq A \subseteq \Theta), \quad \mathcal{PL}^U = Cl(pl_A, \emptyset \subseteq A \subseteq \Theta),$$
$$\mathcal{Q}^U = Cl(Q_A, \emptyset \subseteq A \subseteq \Theta).$$

For a frame $\Theta_2 = \{x, y\}$ of cardinality 2 they form three-dimensional simplices embedded in a four-dimensional Cartesian space:

$$\begin{array}{rcl} \mathcal{B} & = & Cl(b_\emptyset = [1,1,1,1]', b_x = [0,1,0,1]', b_y = [0,0,1,1]', b_\Theta = [0,0,0,1]'); \\ \mathcal{PL} & = & Cl(pl_\emptyset = [0,0,0,0]', pl_x = [0,1,0,1]', pl_y = [0,0,1,1]', pl_\Theta = [0,1,1,1]'); \\ \mathcal{Q} & = & Cl(Q_\emptyset = [1,0,0,0]', Q_x = [1,1,0,0]', Q_y = [1,0,1,0]', Q_\Theta = [1,1,1,1]'). \end{array} \tag{21}$$

We know from Section 4.2.4 that $\mathcal{PL}_2$ and $\mathcal{B}_2$ are congruent. By Equation (21) it follows that

$$\begin{array}{rcccccccl} \|b_\emptyset - b_x\|_2 & = & \|[1,0,1,0]'\|_2 & = & \sqrt{2} & = & \|[0,1,0,1]'\|_2 & = & \|pl_x - pl_\emptyset\|_2, \\ \|b_y - b_\Theta\|_2 & = & \|[0,0,1,0]'\|_2 & = & 1 & = & \|[0,1,0,0]'\|_2 & = & \|pl_\Theta - pl_y\|_2 \end{array}$$

etcetera, and as $\mathcal{B}_2^U$ and $\mathcal{PL}_2^U$ are simplices they are also congruent.

5.1.3 Congruence of plausibility and commonality spaces

A similar result holds for plausibility and commonality spaces. We first need to point out the relationship between the vertices of plausibility and commonality spaces in the unnormalized case, as

$$pl_A = -\sum_{\emptyset \subsetneq B \subseteq A} (-1)^{|B|} b_B$$

while

$$Q_A = \sum_{\emptyset \subseteq B \subseteq A^c} (-1)^{|B|} b_B = \sum_{\emptyset \subsetneq B \subseteq A^c} (-1)^{|B|} b_B + b_\emptyset = -pl_{A^c} + b_\emptyset. \tag{22}$$

Theorem 6. *The 1-dimensional faces $Cl(Q_B, Q_A)$ and $Cl(pl_{B^c}, pl_{A^c})$ of commonality and plausibility spaces respectively are congruent, namely*

$$\|Q_B - Q_A\|_p = \|pl_{B^c} - pl_{A^c}\|_p.$$

Proof. Since $Q_A = b_\Theta - pl_{A^c}$ then

$$Q_A - Q_B = b_\emptyset - pl_{A^c} - b_\emptyset + pl_{B^c} = pl_{B^c} - pl_{A^c}$$

and the two faces are obviously congruent. □

Therefore, the following map between vertices of $\mathcal{PL}^U$ and $\mathcal{Q}^U$

$$Q_A \mapsto pl_{A^c} \tag{23}$$

maps 1-dimensional faces of the commonality space to congruent faces of the plausibility space $Cl(Q_A, Q_B) \mapsto Cl(pl_{A^c}, pl_{B^c})$. The two simplices are congruent. However, (23) clearly acts as a 1-1 application of *unnormalized* categorical commonality and plausibility functions (as the complement of $\emptyset$ is Θ, so that $Q_\Theta \mapsto pl_\emptyset$). Therefore we can only claim that:

Corollary 2. *$\mathcal{Q}^U$ and $\mathcal{PL}^U$ are congruent.*

Of course, in virtue of Corollary 1, we also have that:

Corollary 3. *$\mathcal{Q}^U$ and $\mathcal{B}^U$ are congruent.*

5.1.4 Running example: congruence of $\mathcal{Q}_2$ and $\mathcal{PL}_2$

Let us get back to the binary example: $\Theta_2 = \{x, y\}$. It is easy to see from Figures 2 and 3 that $\mathcal{PL}_2$ and $\mathcal{Q}_2$ are *not* congruent in the case of *normalized* belief functions, as $\mathcal{Q}_2$ is an equilateral triangle with sides of length $\sqrt{2}$, while $\mathcal{PL}_2$ has two sides of length 1.
In the unnormalized case instead, recalling Equation (21),

$$\begin{array}{rclrcl} Q_\Theta - Q_\emptyset & = & [0,1,1,1]', & pl_\Theta - pl_\emptyset & = & [0,1,1,1]' \\ Q_x - Q_y & = & [0,1,-1,0]', & pl_x - pl_y & = & [0,1,-1,0]' \\ Q_x - Q_\Theta & = & [0,0,-1,-1]', & pl_\emptyset - pl_y & = & [0,0,-1,-1]' \end{array} \tag{24}$$

etcetera, confirming that $\mathcal{Q}_2^U$ and $\mathcal{PL}_2^U$ are indeed congruent.

5.2 Point-wise rigid transformation

Belief, plausibility and commonality functions form simplices which can be moved onto each other by means of a rigid transformation, as a reflection of the equivalence of the associated models. It can be interesting to analyze also the geometric behavior of *single* functions, i.e., of the triplet of associated non-additive measures (b, pl_b, Q_b). In binary case (Section 4.2.4) the point-wise geometry of a plausibility vector can be described in terms of a reflection with respect to the probability simplex $\mathcal{P}$. In the general case, as the simplices $\mathcal{B}^U$, $\mathcal{PL}^U$, and $\mathcal{Q}^U$ are all congruent, there must exist an Euclidean transformation $\tau \in E(N)$ mapping each simplex onto one of the others.

5.2.1 Belief and plausibility spaces

In the case of belief and plausibility spaces (in the standard, normalized case) the rigid transformation is clearly encoded by Equation (3): $pl_b(A) = 1 - b(A^c)$. Since $pl_b = \sum_{\emptyset \subsetneq A \subseteq \Theta} pl_b(A) X_A$ Equation (3) implies the following relation

$$pl_b = \mathbf{1} - b^c,$$

where b^c is the unique belief function whose belief values are the same as b's on the complement of each event A: $b^c(A) = b(A^c)$.
As in the normalized case $\mathbf{1} = pl_\Theta$ and $\mathbf{0} = b_\Theta$, the above relation reads as

$$pl_b = \mathbf{1} - b^c = \mathbf{0} + \mathbf{1} - b^c = b_\Theta + pl_\Theta - b^c.$$

Therefore, the segments $Cl(b_\Theta, pl_\Theta)$ and $Cl(b^c, pl_b)$ have the same center of mass, for

$$\frac{pl_b + b^c}{2} = \frac{b_\Theta + pl_\Theta}{2}.$$

In other words:

Theorem 7. *The plausibility vector pl_b associated with a belief function b is the reflection in $\mathbb{R}^{N-2}$ through the segment $Cl(b_\Theta, pl_\Theta) = Cl(\mathbf{0}, \mathbf{1})$ of the "complement" belief function b^c.*

Geometrically, b^c is obtained from b by means of another reflection (by swapping the coordinates associated with the axes X_A and X_{A^c}), so that the desired rigid transformation is completely determined.
Figure 4 illustrates the nature of the transformation, and its instantiation in the binary case for normalized belief functions. In the case of unnormalized belief functions ($b_\emptyset = \mathbf{1}$, $pl_\emptyset = \mathbf{0}$) we have

$$pl_b = pl_\emptyset + b_\emptyset - b^c,$$

i.e., pl_b is the reflection of b^c through the segment $Cl(b_\emptyset, pl_\emptyset) = Cl(\mathbf{0}, \mathbf{1})$.

5.2.2 Commonality and plausibility spaces

The form of the point-wise transformation is also quite simple in the case of the pair $(\mathcal{PL}^U, \mathcal{Q}^U)$. We can indeed use Equation (22), getting

$$Q_b = \sum_{\emptyset \subseteq A \subseteq \Theta} m_b(A) Q_A = \sum_{\emptyset \subseteq A \subseteq \Theta} m_b(A)(b_\emptyset - pl_{A^c}) = b_\emptyset - \sum_{\emptyset \subseteq A \subseteq \Theta} m_b(A) pl_{A^c} =$$

$= b_\emptyset - pl_{b^{m^c}}$, where b^{m^c} is the unique belief function whose b.p.a. is $m_{b^{m^c}}(A) = m_b(A^c)$. But then, since $pl_\emptyset = \mathbf{0} = [0, \cdots, 0]'$ for unnormalized belief functions (remember the binary example), we can rewrite the above equation as

$$Q_b = pl_\emptyset + b_\emptyset - pl_{b^{m^c}}.$$

In conclusion,

Theorem 8. *The commonality vector associated with a belief function b is the reflection in $\mathbb{R}^N$ through the segment $Cl(pl_\emptyset, b_\emptyset) = Cl(\mathbf{0}, \mathbf{1})$ of the plausibility vector $pl_{b^{m^c}}$ associated with the belief function b^{m^c}.*

In this case, though, b^{m^c} is obtained from b by swapping the coordinates with respect to the base $\{b_A, \emptyset \subseteq A \subseteq \Theta\}$. A pictorial representation for the binary case (similar to Figure 4) is more difficult in this case as $\mathbb{R}^4$ is involved. It is natural to stress the analogy between the two rigid transformations

$$\tau_{\mathcal{B}^U \mathcal{PL}^U} : \mathcal{B}^U \to \mathcal{PL}^U, \quad \tau_{\mathcal{PL}^U \mathcal{Q}^U} : \mathcal{PL}^U \to \mathcal{Q}^U$$

mapping respectively an unnormalized belief function onto the corresponding plausibility, and an unnormalized pl.f. onto the corresponding commonality function:

$$\begin{array}{lllllll} \tau_{\mathcal{B}^U \mathcal{PL}^U} & : & b & \stackrel{b(A) \mapsto b(A^c)}{\longrightarrow} & b^c & \stackrel{refl.\ through\ Cl(\mathbf{0},\mathbf{1})}{\longrightarrow} & pl_b \\ \tau_{\mathcal{PL}^U \mathcal{Q}^U} & : & pl_b & \stackrel{m_b(A) \mapsto m_b(A^c)}{\longrightarrow} & b^{m^c} & \stackrel{refl.\ through\ Cl(\mathbf{0},\mathbf{1})}{\longrightarrow} & Q_b. \end{array}$$

They both have the form of a sequence of two reflections: a swap of the axes of the reference frame $\{X_A\}$ ($\{b_A\}$) induced by set-theoretic complement, plus a reflection with respect to the center of the segment $Cl(\mathbf{0}, \mathbf{1})$.

6 Application to probability transformation

The three equivalent descriptions of the evidence represented by belief, plausibility and commonality functions possess the same combinatorial structure of sum functions. As such, they admit Moebius inverses called basic probability, plausibility, and commonality assignments.
Immediately, then, all the aggregation operators that apply to belief functions through their Moebius inverse (the b.p.a.) can be extended to the combination of plausibility or commonality functions (Section 6.1). As an entire class of probabilistic approximations of belief functions relates to such operators (and to Dempster's rule [12, 11] in particular), this has interesting consequences on the problem of transforming a complex object such as a belief function into a standard probability distribution [42, 14, 18, 37, 25, 2, 32, 38, 3].
As significant examples we discuss in more detail the properties enjoyed by two probability transformations called "relative belief of singletons" [6] (Section 6.2) and "intersection probability" [5] (Section 6.3). We will make use of the vectorial representation of belief and plausibility functions recalled in Section 4.

6.1 Combination of plausibility functions

Historically, different combination rules have been proposed to merge the evidence carried by different belief functions, starting with Dempster's original proposal [12].

Definition 1. *The* orthogonal sum *or* Dempster's sum *of two belief functions* $b_1, b_2 : 2^\Theta \to [0, 1]$ *is a new b.f.* $b_1 \oplus b_2 : 2^\Theta \to [0, 1]$ *with basic probability* *assignment:*

$$m_{b_1 \oplus b_2}(A) = \frac{\displaystyle\sum_{B \cap C = A} m_{b_1}(B) m_{b_2}(C)}{\displaystyle\sum_{B \cap C \neq \emptyset} m_{b_1}(B) m_{b_2}(C)} \qquad A \subseteq \Theta. \tag{25}$$

We denote by $k(b_1, b_2)$ the denominator of (25).
Other operators have been later brought forward, notably in the context of the Transferable Belief Model [34].

Definition 2. *The* conjunctive combination *of two belief functions* $b_1, b_2 : 2^\Theta \to [0,1]$ *is a new b.f.* $b_1 \cap b_2 : 2^\Theta \to [0,1]$ *with basic probability assignment:*

$$m_{b_1 \cap b_2}(A) = \sum_{B \cap C = A} m_{b_1}(B) m_{b_2}(C). \tag{26}$$

Their disjunctive combination *is the unique belief function* $b_1 \cup b_2 : 2^\Theta \to [0,1]$ *with b.p.a.:*

$$m_{b_1 \cup b_2}(A) = \sum_{B \cup C = A} m_{b_1}(B) m_{b_2}(C). \tag{27}$$

Obviously Dempster's sum, disjunctive and conjunctive rules can be applied to any pair ς_1, ς_2 of (normalized) sum functions. We just need to apply (25), (27), or (26) to the corresponding Moebius inverses m_{ς_1}, m_{ς_2}.
In particular, they can be applied to plausibility functions through their basic plausibility assignments (5).

6.2 Dual properties of relative belief and plausibility of singletons

One interesting approach to the probability transformation problem seeks approximations which enjoy commutativity properties with respect to a specific combination rule, in particular Dempster's sum [12, 11] (Equation 25). Voorbraak has been the first to explore this direction, proposing the adoption of the so-called *relative plausibility of singletons* [38]:

$$\tilde{pl}_b(x) = \frac{pl_b(x)}{\displaystyle\sum_{y \in \Theta} pl_b(y)}. \tag{28}$$

It can indeed be proven that (28) is strictly related to Dempster's rule, as the following propositions ([38],[3]) demonstrate.

Proposition 2. *The relative plausibility of singletons* $\tilde{pl}_b$ *perfectly represent the corresponding belief function* b *when combined with any probability measure* $p \in \mathcal{P}$ *through Dempster's rule, namely:*

$$b \oplus p = \tilde{pl}_b \oplus p, \qquad \forall p \in \mathcal{P}.$$

Proposition 3. *If $b = b_1 \oplus \cdots \oplus b_m$ then $\tilde{pl}_b = \tilde{pl}_{b_1} \oplus \cdots \oplus \tilde{pl}_{b_m}$. In other words, Dempster's sum and relative plausibility commute.*

A dual transformation called *relative belief of singletons*

$$\tilde{b}(x) \doteq \frac{b(x)}{\sum_{y\in\Theta} b(y)} \tag{29}$$

can be introduced by simply swapping plausibility values for belief values. It can proven that, as a consequence, (29) meets properties which are the dual of those expressed by Propositions 2 and 3. However, as such properties involve the combination of plausibility functions, they rely on the existence of basic plausibility assignments. To see this we first need to recall a useful result on Dempster's sum of affine combinations [8].

Lemma 2. *The orthogonal sum $b \oplus \sum_i \alpha_i b_i$, $\sum_i \alpha_i = 1$ of a belief function b with any*[3] *affine combination of belief functions can be written as*

$$b \oplus \sum_i \alpha_i b_i = \sum_i \gamma_i (b \oplus b_i) \tag{30}$$

where

$$\gamma_i = \frac{\alpha_i k(b, b_i)}{\sum_j \alpha_j k(b, b_j)} \tag{31}$$

and $k(b, b_i)$ is the normalization factor of the combination of b and b_i.

Proposition 4. *The relative belief of singletons $\tilde{b}$ perfectly represents the corresponding plausibility function pl_b when combined with any probability measure through (extended) Dempster's rule, namely:*

$$\tilde{b} \oplus p = pl_b \oplus p$$

for each Bayesian belief function $p \in \mathcal{P}$.

Proof. By Equation (17) we can express each plausibility function as an affine combination of categorical b.f.s, whose coefficients are given by its basic plausibility assignment:

$$pl_b = \sum_{A\subseteq\Theta} \mu(A) b_A.$$

We can then apply the commutativity property (30) of Lemma 2, obtaining

$$pl_b \oplus p = \sum_{A\subseteq\Theta} \nu(A) p \oplus b_A \tag{32}$$

where

$$\nu(A) = \frac{\mu_b(A) k(p, b_A)}{\displaystyle\sum_{B\subseteq\Theta} \mu_b(B) k(p, b_B)}, \qquad p \oplus b_A = \frac{\displaystyle\sum_{x\in A} p(x) b_x}{k(p, b_A)}$$

[3] In fact the collection $\{b_i, i\}$ is required to include *at least* a belief function combinable with b [8].

with $k(p, b_A) = \sum_{x\in A} p(x)$. Once replaced these expressions in (32) we get

$$\begin{array}{lll} pl_b \oplus p & = & \dfrac{\displaystyle\sum_{A\subseteq\Theta} \mu_b(A)\Big(\sum_{x\in A} p(x) b_x\Big)}{\displaystyle\sum_{B\subseteq\Theta} \mu_b(B)\Big(\sum_{y\in B} p(y)\Big)} = \dfrac{\displaystyle\sum_{x\in\Theta} p(x)\Big(\sum_{A\supseteq\{x\}} \mu_b(A)\Big) b_x}{\displaystyle\sum_{y\in\Theta} p(y)\Big(\sum_{B\supseteq\{y\}} \mu_b(B)\Big)} \\ & = & \dfrac{\displaystyle\sum_{x\in\Theta} p(x) m_b(x) b_x}{\displaystyle\sum_{y\in\Theta} p(y) m_b(y)}, \end{array}$$

by Theorem 2. But this is exactly $\tilde{b} \oplus p$, as a direct application of Dempster's rule (25) shows. □

Proposition 5. *The relative belief operator*

$$\begin{array}{rccc} \tilde{b} & : & \mathcal{PL} & \rightarrow & \mathcal{P} \\ & & pl_b & \mapsto & \tilde{b}[pl_b], \end{array}$$

mapping a plausibility function to the corresponding relative belief of singletons (29), commutes with Dempster's combination of plausibility functions:

$$\tilde{b}[pl_1 \oplus pl_2] = \tilde{b}[pl_1] \oplus \tilde{b}[pl_2].$$

Proof. The basic plausibility assignment of $pl_1 \oplus pl_2$ is, according to (25),

$$\mu_{pl_1 \oplus pl_2}(A) = \frac{1}{k(pl_1, pl_2)} \sum_{X\cap Y = A} \mu_1(X)\mu_2(Y).$$

Thus, according to Theorem 2, the corresponding relative belief of singletons $\tilde{b}[pl_1 \oplus pl_2](x)$ is proportional to $m_{pl_1\oplus pl_2}(x) =$

$$= \sum_{A\supseteq\{x\}} \mu_{pl_1\oplus pl_2}(A) = \frac{\displaystyle\sum_{A\supseteq\{x\}} \sum_{X\cap Y=A} \mu_1(X)\mu_2(Y)}{k(pl_1, pl_2)} = \frac{\displaystyle\sum_{X\cap Y\supseteq\{x\}} \mu_1(X)\mu_2(Y)}{k(pl_1, pl_2)}, \tag{33}$$

where $m_{pl_1\oplus pl_2}(x)$ denotes the Moebius inverse of the normalized sum function equal to the plausibility function $pl_1 \oplus pl_2$.
On the other hand, as $\sum_{X\supseteq\{x\}} \mu_b(X) = m_b(x)$ by Equation (10),

$$\tilde{b}[pl_1](x) \propto m_1(x) = \sum_{X\supseteq\{x\}} \mu_1(X), \qquad \tilde{b}[pl_2](x) \propto m_2(x) = \sum_{X\supseteq\{x\}} \mu_2(X).$$

Their Dempster's combination is therefore

$$(\tilde{b}[pl_1] \oplus \tilde{b}[pl_2])(x) \propto \Big(\sum_{X\supseteq\{x\}} \mu_1(X)\Big)\Big(\sum_{Y\supseteq\{x\}} \mu_2(Y)\Big) = \sum_{X\cap Y\supseteq\{x\}} \mu_1(X)\mu_2(Y),$$

and by normalizing we get (33). □

Generally speaking, each time it is necessary or convenient to compute combinations *of plausibility functions* the usefulness of basic plausibility assignments becomes apparent. In particular, this is true when exploring properties obtained by duality, i.e., by exchanging the role of belief and plausibility functions in existing statements. Another example is provided by probability transformations inherently associated with the couple (b, pl_b). This is the case of the "intersection probability" [5].

6.3 The discovery of the intersection probability

Each pair (b, pl_b) of belief/plausibility functions determines a line that, in the binary case, crosses the set $\mathcal{P}$ of probability distributions or "Bayesian" belief functions (Figure 2). In that case the intersection coincides with Smets' pignistic transformation [32]:

$$BetP[b](x) = \sum_{A \supseteq \{x\}} \frac{m_b(A)}{|A|}.$$

This is not the case for general frames of discernment. It is true, however, that the line $a(b, pl_b)$ intersects the set $\mathcal{P}'$ of Bayesian normalized sum functions (i.e., n.s.f.s ς such that $\sum_x \varsigma(x) = 1$) in:

$$\varsigma[b] = b + \beta[b](pl_b - b).$$

Such intersection is uniquely associated with a probability transformation called *intersection probability* [5]

$$p[b](x) = m_b(x) + \beta[b](pl_b(x) - m_b(x)), \tag{34}$$

where $\beta[b] = \frac{1 - \sum_x m_b(x)}{\sum_x (pl_b(x) - m_b(x))}$ is a scalar function of b.

The rationale of the intersection probability is clearer in the framework of probability intervals [9]. A belief function b determines an interval of admissible probability values $b(x) \leq p(x) \leq pl_b(x)$ for all the elements $x \in \Theta$ of the domain. As there is no reason to favor any of these elements over the others, it is sensible to seek a probability transformation of the form

$$p(x) = b(x) + \alpha(pl_b(x) - b(x))$$

assigning the same, constant ratio α of the interval width $pl_b(x) - b(x)$ to all the elements x of Θ. Such transformation is indeed given by the intersection probability.

A justification for the name of this function $p[b]$ comes from the following result [4], obtained by expressing pl_b as a sum function (6).

Proposition 6. *The orthogonal sums of* $p[b]$ *and* $\varsigma[b]$ *with any arbitrary probability function* $p \in \mathcal{P}$ *coincide:*

$$p[b] \oplus p = \varsigma[b] \oplus p \qquad \forall p \in \mathcal{P}.$$

Proof. (sketch) Crucial to the proof of Proposition 6 is being able to compute $\varsigma[b] \oplus p$ [4]. Applying Equation (30) to $\varsigma \oplus p$ yields $\varsigma \oplus p =$

$$= \big[\beta[b]pl_b + (1-\beta[b])b\big] \oplus p = \frac{\beta[b]k(p,pl_b)pl_b \oplus p + (1-\beta[b])k(p,b)b \oplus p}{\beta[b]k(p,pl_b) + (1-\beta[b])k(p,b)}$$

where

$$k(p,b) = \sum_{x\in\Theta} p(x)\bigg(\sum_{A\supseteq\{x\}} m_b(A)\bigg) = \sum_{x\in\Theta} p(x)pl_b(x)$$

is the normalization factor of the standard Dempster's combination $b \oplus p$ of b and p, while

$$k(p,pl_b) = \sum_{x\in\Theta} p(x)\bigg(\sum_{A\supseteq\{x\}} \mu_b(A)\bigg) = \sum_{x\in\Theta} p(x)m_b(x)$$

(thanks to Theorem 2) is the normalization factor for $pl_b \oplus p$. A direct application of Dempster's rule to $p[b] \oplus b$ proves that the latter coincides with $\varsigma[b] \oplus p$. □

Note that $\varsigma[b]$ is a normalized sum function, to which Dempster's rule can be applied in virtue of the argument of Section 6.1.
Even though $p[b]$ is *not* the actual intersection between the line $a(b, pl_b)$ and the region of Bayesian n.s.f.s (which is $\varsigma[b]$), it behaves exactly like $\varsigma[b]$ when combined with a probability distribution.

7 Comments and conclusions

In this paper we proved that plausibility and commonality functions share with belief functions the form of sum functions on the partially ordered set 2^Θ. We introduced for both functions their Moebius inverses, that we called basic plausibility and commonality assignments, and used them to introduce equivalent alternative combinatorial formulations of the theory of evidence.
Belief, plausibility, and commonality functions (even though they are equivalent representations of the same evidence) form a hierarchy of sum functions. Their Moebius inverses meet both normalization and positivity constraints (b.p.a.), just the normalization constraint (b.pl.a.), or none of them (b.comm.a.), respectively.

Quantity	Moebius inverse	
belief function	b.p.a.	non-negative n.s.f.
plausibility function	b.pl.a.	n.s.f.
commonality function	b.comm.a.	sum function.

Nevertheless, they possess a similar simplicial geometry. We analyzed the global structure of such simplices, proving in particular their congruence, and ventured into the description of the point-wise geometry of the triplet (b, pl_b, Q_b).
The alternative models introduced here can be successfully applied to problems like probabilistic transformation or canonical decomposition.

Put in perspective, these results are just a symptom of the strict relationship between combinatorics and subjective probability [17, 16, 21]. The geometric language in which those results are expressed hints to points of contact with the field of geometric probability [22] (which studies invariant measures on sets of geometric objects and relates them to additive probability measures) that could lead to a fertile contamination of the two fields.

Appendix: Proof of Lemma 1

We first need to notice that a categorical belief function can be expressed as $b_A = \sum_{C \supseteq A} X_C$. The entry of the vector b_A associated with the event $\emptyset \subsetneq B \subsetneq \Theta$ is by definition:

$$b_A(B) = \begin{cases} 1 & B \supseteq A \\ 0 & B \not\supset A. \end{cases}$$

As $X_C(B) = 1$ iff $B = C$, 0 otherwise, the corresponding entry of the vector $\sum_{C \supseteq A} X_C$ is also:

$$\sum_{C \supseteq A} X_C(B) = \begin{cases} 1 & B \supseteq A \\ 0 & B \not\supset A. \end{cases}$$

Therefore, if (16) is true we have that

$$b_A = \sum_{C \supseteq A} X_C = \sum_{C \supseteq A} \sum_{B \supseteq C} b_B (-1)^{|B \setminus C|} = \sum_{B \supseteq A} b_B \Big(\sum_{A \subseteq C \subseteq B} (-1)^{|B \setminus C|} \Big).$$

Let us then consider the factor $\sum_{A \subseteq C \subseteq B} (-1)^{|B \setminus C|}$. When $A = B$, $C = A = B$ and the coefficient becomes 1.
On the other side, when $B \neq A$ we have that

$$\sum_{A \subseteq C \subseteq B} (-1)^{|B \setminus C|} = \sum_{D \subseteq B \setminus A} (-1)^D = 0$$

by Newton's binomial ($\sum_{k=0}^{n} 1^{n-k} (-1)^k = [1 + (-1)]^n = 0$). Hence $b_A = b_A$.

References

[1] M. Aigner, *Combinatorial theory*, Classics in Mathematics, Springer, New York, 1979.

[2] M. Bauer, *Approximation algorithms and decision making in the Dempster-Shafer theory of evidence–an empirical study*, International Journal of Approximate Reasoning **17** (1997), no. 2-3, 217–237.

[3] B. R. Cobb and P. P. Shenoy, *A comparison of bayesian and belief function reasoning*, Information Systems Frontiers **5** (2003), no. 4, 345–358.

[4] F. Cuzzolin, *Rationale and properties of the intersection probability*, submitted to Artificial Intelligence Journal (2007).

[5] ______, *Two new Bayesian approximations of belief functions based on convex geometry*, IEEE Transactions on Systems, Man and Cybernetics part B **37** (2007), no. 4, 993–1008.

[6] ______, *Dual properties of the relative belief of singletons*, Proceedings of the Tenth Pacific Rim Conference on Artificial Intelligence (PRICAI'08), Hanoi, Vietnam, December 15-19 2008, 2008.

[7] ______, *A geometric approach to the theory of evidence*, IEEE Transactions on Systems, Man and Cybernetics part C **38** (2008), no. 4, 522–534.

[8] ______, *Geometry of Dempster's rule of combination*, IEEE Transactions on Systems, Man and Cybernetics part B **34** (April 2004), no. 2, 961–977.

[9] L. de Campos, J. Huete, and S. Moral, *Probability intervals: a tool for uncertain reasoning*, Int. J. Uncertainty Fuzziness Knowledge-Based Syst. **1** (1994), 167–196.

[10] A.P. Dempster, *Upper and lower probability inferences based on a sample from a finite univariate population*, Biometrika **54** (1967), 515–528.

[11] ______, *A generalization of Bayesian inference*, Journal of the Royal Statistical Society, Series B **30** (1968), 205–247.

[12] ______, *Upper and lower probabilities generated by a random closed interval*, Annals of Mathematical Statistics **39** (1968), 957–966.

[13] ______, *Upper and lower probabilities inferences for families of hypothesis with monotone density ratios*, Annals of Mathematical Statistics **40** (1969), 953–969.

[14] T. Denoeux, *Inner and outer approximation of belief structures using a hierarchical clustering approach*, Int. Journal of Uncertainty, Fuzziness and Knowledge-Based Systems **9** (2001), no. 4, 437–460.

[15] R. Fagin and J.Y. Halpern, *Uncertainty, belief and probability*, Proc. Intl. Joint Conf. in AI (IJCAI-89), 1988, pp. 1161–1167.

[16] M. Grabisch, *The Moebius transform on symmetric ordered structures and its application to capacities on finite sets*, Discrete Mathematics **287** (2004), no. 1–3, 17–34.

[17] ______, *Belief functions on lattices*, Int. J. of Intelligent Systems (2006).

[18] R. Haenni and N. Lehmann, *Resource bounded and anytime approximation of belief function computations*, International Journal of Approximate Reasoning **31** (2002), no. 1-2, 103–154.

[19] J.Y. Halpern, *Reasoning about uncertainty*, MIT Press, 2003.

[20] H.T. Hestir, H.T. Nguyen, and G.S. Rogers, *A random set formalism for evidential reasoning*, Conditional Logic in Expert Systems, North Holland, 1991, pp. 309–344.

[21] A. Honda and M. Grabisch, *Entropy of capacities on lattices and set systems*, Information Science (2006).

[22] D.A. Klain and G.-C. Rota, *Introduction to geometric probability*, Cambridge University Press, 1997.

[23] J. Kohlas, *Mathematical foundations of evidence theory*, Mathematical Models for Handling Partial Knowledge in Artificial Intelligence (G. Coletti, D. Dubois, and R. Scozzafava, eds.), Plenum Press, 1995, pp. 31–64.

[24] I. Kramosil, *Measure-theoretic approach to the inversion problem for belief functions*, Fuzzy Sets and Systems **102** (1999), no. 3, 363–369.

[25] J.D. Lowrance, T.D. Garvey, and T.M. Strat, *A framework for evidential-reasoning systems*, Proceedings of the National Conference on Artificial Intelligence (American Association for Artificial Intelligence, ed.), 1986, pp. 896–903.

[26] H.T. Nguyen, *On random sets and belief functions*, J. Mathematical Analysis and Applications **65** (1978), 531–542.

[27] E.H. Ruspini, *Epistemic logics, probability and the calculus of evidence*, Proc. 10th Intl. Joint Conf. on AI (IJCAI-87), 1987, pp. 924–931.

[28] G. Shafer, *A mathematical theory of evidence*, Princeton University Press, 1976.

[29] ———, *Allocations of probability*, Annals of Probability **7** (1979), no. 5, 827–839.

[30] ———, *Perspectives on the theory and practice of belief functions*, International Journal of Approximate Reasoning **4** (1990), 323–362.

[31] G. Shafer and N. Vovk, *Probability and finance: It's only a game!*, Wiley, New York, 2001.

[32] Ph. Smets, *Belief functions versus probability functions*, Uncertainty and Intelligent Systems (Saitta L. Bouchon B. and Yager R., eds.), Springer Verlag, 1988, pp. 17–24.

[33] ———, *The nature of the unnormalized beliefs encountered in the transferable belief model*, Proceedings of UAI'92 (San Mateo, CA), Morgan Kaufmann, 1992, pp. 292–29.

[34] ———, *Belief functions : the disjunctive rule of combination and the generalized Bayesian theorem*, International Journal of Approximate Reasoning **9** (1993), 1–35.

[35] ———, *The canonical decomposition of a weighted belief*, Proceedings of the International Joint Conference on AI, IJCAI'95, Montréal, Canada, 1995, pp. 1896–1901.

[36] Ph. Smets and R. Kennes, *The transferable belief model*, Artificial Intelligence **66** (1994), no. 2, 191–234.

[37] B. Tessem, *Approximations for efficient computation in the theory of evidence*, Artificial Intelligence **61** (1993), no. 2, 315–329.

[38] F. Voorbraak, *A computationally efficient approximation of Dempster-Shafer theory*, International Journal on Man-Machine Studies **30** (1989), 525–536.

[39] P. Walley, *Statistical reasoning with imprecise probabilities*, Chapman and Hall, New York, 1991.

[40] ———, *Towards a unified theory of imprecise probability*, International Journal of Approximate Reasoning **24** (2000), no. 2-3, 125–148.

[41] Z. Wang and G.J. Klir, *Choquet integrals and natural extensions of lower probabilities*, International Journal of Approximate Reasoning **16** (1997), no. 2, 137–147.

[42] A. Ben Yaghlane, T. Denoeux, and K. Mellouli, *Coarsening approximations of belief functions*, Proceedings of ECSQARU'2001 (S. Benferhat and P. Besnard, eds.), 2001, pp. 362–373.

Figure captions

Figure 1: The belief function of the example 2.1 has two focal elements, $\{x\}$ and Θ.

Figure 2: Geometry of belief and plausibility spaces in the binary case.

Figure 3: Commonality space in the binary case.

Figure 4: Rigid transformation mapping b onto pl_b in the normalized case. In the binary case the middle point of the segment $Cl(\mathbf{0}, \mathbf{1})$ is the mean probability $\overline{\mathcal{P}}$.

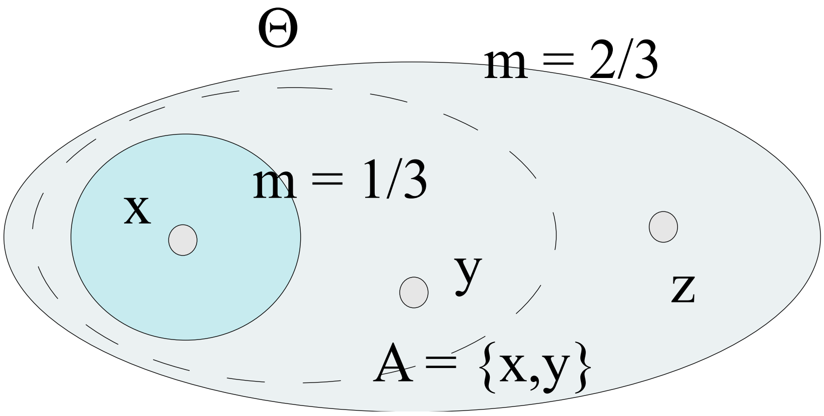
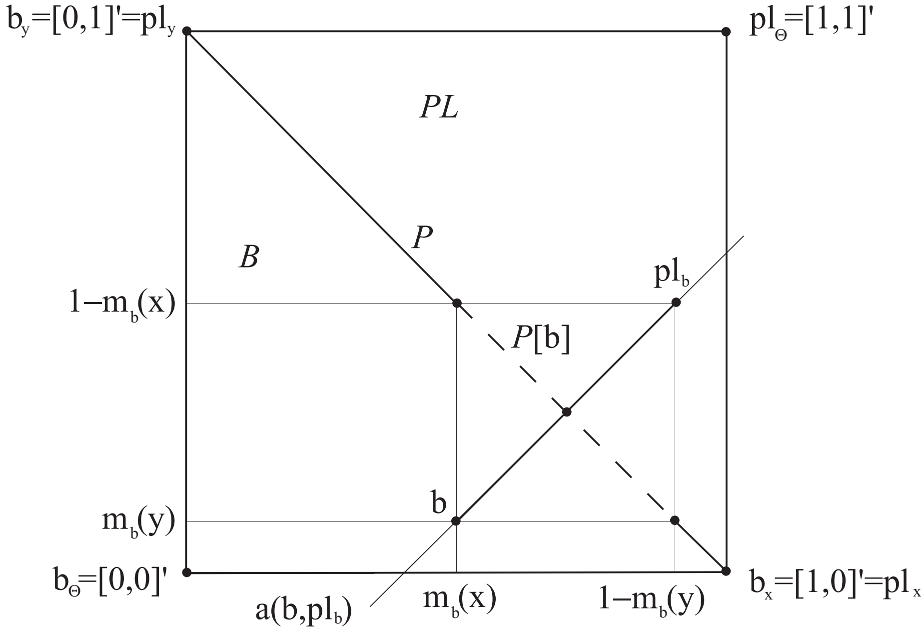


Figure 1:



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pl_Θ=[1,1]'
PL
P
B
pl_b
1–m_b(x)
P[b]
b
m_b(y)
b_Θ=[0,0]'
b_x=[1,0]'=pl_x
a(b,pl_b)
m_b(x)
1–m_b(y)

Figure 2:

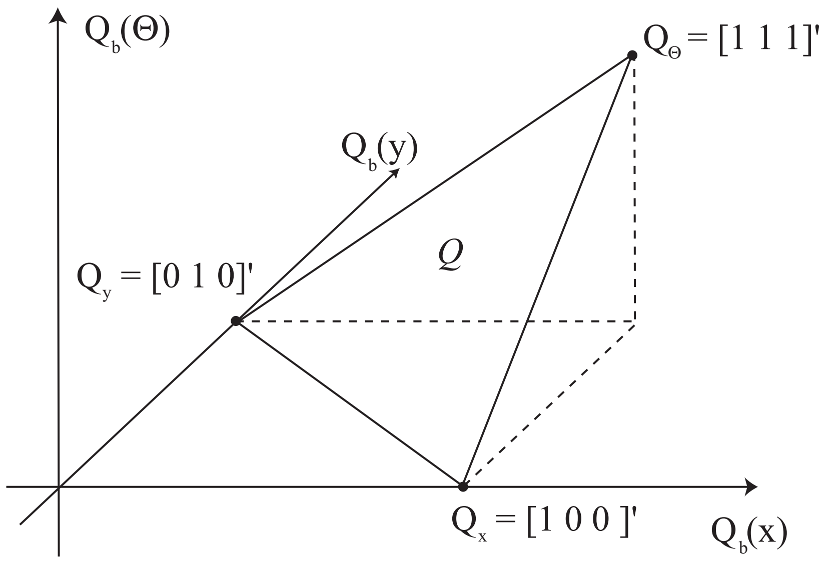

Figure 3:

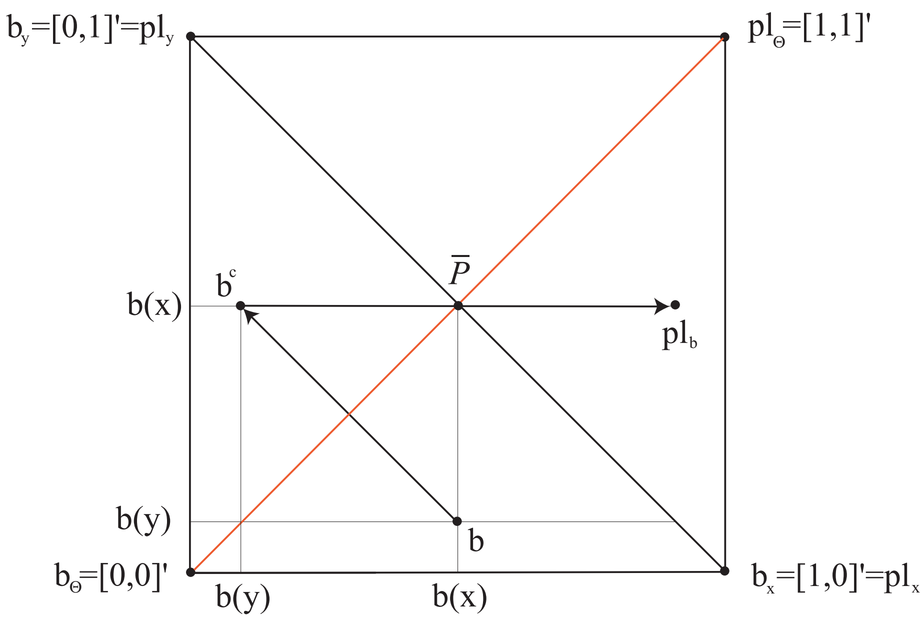
b_y=[0,1]'=pl_y
pl_Θ=[1,1]'
$\overline{P}$
b^c
b(x)
pl_b
b(y)
b
b_Θ=[0,0]'
b(y)
b(x)
b_x=[1,0]'=pl_x

Figure 4: