The geometry of consonant belief functions: Simplicial complexes of necessity measures

Fabio Cuzzolin

Department of Computing, School of Technology
Oxford Brookes University, Oxford, UK.

Abstract

In this paper we extend the geometric approach to the theory of evidence in order to include the class of necessity measures, represented on a finite domain of "frame" by consonant belief functions (b.f.s). The correspondence between chains of subsets and convex sets of b.f.s is studied and its properties analyzed, eventually yielding an elegant representation of the region of consonant belief functions in terms of the notion of "simplicial complex". In particular we focus on the set of outer consonant approximations of a belief function, showing that for each maximal chain of subsets these approximations form a polytope. The maximal such approximation with respect to the weak inclusion relation between b.f.s is one of the vertices of this polytope, and is generated by a permutation of the elements of the frame.

Key words: Theory of evidence, geometric approach, necessity measures, consonant belief functions, simplicial complex, outer consonant approximations.
PACS:

1 Introduction

Uncertainty measures have a mayor role in fields like artificial intelligence, where problems involving formalized reasoning or machine learning are common. Many engineering tasks require making decisions under scarce information, and are fertile ground for applications of uncertainty theory. Autonomous navigation [1], database management [2], computer vision [3,4] provide significant examples.

Email address: `fabio.cuzzolin@brookes.ac.uk` (Fabio Cuzzolin).
URL: `perception.inrialpes.fr/People/~cuzzolin` (Fabio Cuzzolin).

The theory of evidence (ToE) [5] is one the most popular approaches to uncertainty description, as a natural extension of the classical Bayesian formalism. In the ToE probabilities are replaced by *belief functions* (b.f.s), which assign values between 0 and 1 to subsets of the sample space instead of single elements. Bayes' rule is also replaced by a more general operator called *Dempster's sum* [6] which governs the combination of two or more belief functions. In a series of recent works [7,8] we proposed a geometric interpretation of the theory of evidence in which belief functions are represented as points of a simplex called *belief space* [8]. As a belief function $b : 2^\Theta \rightarrow [0,1]$ defined on the power set 2^Θ of a finite domain Θ is completely specified by its $N-2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$, $N \doteq 2^{|\Theta|}$, it can be represented as a point of $\mathbb{R}^{N-2}$. Another attractive feature of the theory of evidence is the fact that, at least in the case of finite domains, it includes both possibility theory [9] and the theory of fuzzy sets as special cases.
In particular, it is well known that *necessity measures*, i.e., measures of the form $Nec(A) = 1 - Pos(A^c)$, $A \subseteq \Theta$ where Pos is a possibility measure have counterparts in the theory of evidence as *consonant* b.f.s, i.e. belief functions whose focal elements are nested [5].

In this paper we then move forward to analyze the convex geometry of consonant belief functions (co.b.f.s), as a first step towards a unified geometric picture of a wider class of uncertainty measures.
We show that consonant b.f.s are in correspondence with chains of subsets of their domain, and are hence located in a collection of convex regions of the belief space which has the form of a *simplicial complex*, i.e. a structured collection of simplices. This approach, on one side, provides a useful visualization tool which can be used to stimulate conjectures on the properties of the entities of interest. On the other side, it generates new problems and allows to look at known problems from a different perspective.
We illustrate this in the second part of the paper, in which the problem of approximating a belief measure with a necessity measure [10–13] is considered from a geometric point of view. In particular we focus on *outer consonant approximations*, showing that they live on a collection of polytopes associated with all possible maximal chains of focal elements. For a given chain the maximal such approximation is a vertex of the corresponding polytope, is generated by a permutation of the elements of the frame, and coincides with the lower chain measure associated with the original belief function.

1.1 Previous work

Set functions, i.e. functions $\mu : 2^\Theta \rightarrow [0,1]$ s.t. $\mu(\emptyset) = 0$, $\mu(\Theta) = 1$ are a widely studied subject in uncertainty theory, of which belief functions are a special case [14]. In this context a representation of set functions in terms of points of

a vector space has been introduced e.g. in [15,16]. Similar methods have been applied by the author more specifically to the theory of evidence [8]. Other authors have recently worked on the geometry of uncertainty measures. P. Black used shapes of geometric loci to give a direct visualization of belief functions and other classes of monotone capacities [17,18], while Ha and Haddawy [19] presented the interval generalization of the probability cross-product operator, called convex-closure (cc) operator.
On the other side many authors, like Yager [20] and Romer [21] amongst others, have studied the connection between fuzzy numbers and Dempster-Shafer theory. Klir *et al.* published an excellent discussion [22] on the relations among fuzzy and belief measures and possibility theory. Heilpern [23] also presented the theoretical background of fuzzy numbers connected with the possibility and Dempster-Shafer theories, describing some types of representation of fuzzy numbers and studying the notions of distance and order between fuzzy numbers based on these representations. Caro and Nadjar [24], instead, suggested a generalization of the Dempster-Shafer theory to a fuzzy valued measure. The links between transferable belief model and possibility theory have been briefly investigated by Ph. Smets in [25], while Dubois and Prade [10] have worked extensively on consonant approximations of belief functions. Their work has been later considered in [11,26].

1.2 Paper outline

We will first review the basic notions of theory of evidence and possibility theory (Section 2), stressing the relation between consonant belief functions and necessity measures. After recalling in Section 3 the geometric approach to the ToE, we will study the geometry of the space of consonant belief functions, or *consonant subspace* $\mathcal{CO}$ (Section 4). After observing the correspondence between co.b.f.s and maximal chains of events, we will look for useful intuitions by studying the case of ternary frames, and prove that the consonant subspace has the form of a *simplicial complex* [27]. In Section 5 we will investigate in more detail the convex geometry of the components of $\mathcal{CO}$, proving that they are all congruent to each other, and can be decomposed into faces which are right triangles.
In the second part of the paper (Section 6) we will discuss the consonant approximation problem in the framework of the consonant complex. Starting from the simple binary case we will prove that for each maximal chain $\mathcal{C}$ of focal elements the set of outer consonant approximations $\mathcal{O}_\mathcal{C}[b]$ of a b.f. b is not only convex but it forms a polytope. Maximal outer approximations are also investigated, as we prove that for each chain they coincide with both the vertex of $\mathcal{O}_\mathcal{C}[b]$ generated by a permutation of the elements of the frame and the lower chain measure associated with b.
To improve the readability of the paper several major proofs are collected in

an Appendix.

2 Between evidence and possibility: consonant belief functions

In the *theory of evidence* [6,5] subjective probability is represented by *belief functions* (b.f.s) rather than Bayesian mass distributions, assigning probability values to *sets* of possibilities rather than single events. A *basic probability assignment* (b.p.a.) over a finite set *(frame of discernment [5])* Θ is a function $m : 2^\Theta \rightarrow [0,1]$ on its power set $2^\Theta = \{A \subseteq \Theta\}$ such that $m(\emptyset) = 0$, $\sum_{A\subseteq\Theta} m(A) = 1$, and $m(A) \geq 0 \ \forall A \subseteq \Theta$. Subsets of Θ associated with non-zero values of m are called *focal elements* (f.e.s).
The *belief function* $b : 2^\Theta \rightarrow [0,1]$ associated with a basic probability assignment m on Θ is defined as: $b(A) = \sum_{B\subseteq A} m(B)$.
Conversely, the unique basic probability assignment m_b associated with a given belief function b is given by the *Moebius inversion formula*

$$m_b(A) = \sum_{B\subseteq A} (-1)^{|A-B|} b(B) \tag{1}$$

so that there is a 1-1 correspondence between the two set functions $m_b \leftrightarrow b$.
A dual mathematical representation of the evidence encoded by a belief function b is the *plausibility function* (pl.f.) $pl_b : 2^\Theta \rightarrow [0,1]$, $A \mapsto pl_b(A)$ where the plausibility value $pl_b(A)$ of an event A is

$$pl_b(A) \doteq 1 - b(A^c) = 1 - \sum_{B\subseteq A^c} m_b(B) = \sum_{B\cap A\neq\emptyset} m_b(B) \geq b(A) \tag{2}$$

and expresses the amount of evidence *not against* A.
In the theory of evidence a probability function is a special belief function which assigns non-zero masses to singletons only (*Bayesian* b.f.): $m_b(A) = 0$ $|A| > 1$. At the opposite of Bayesian b.f.s stand *consonant* belief functions.

Definition 1 *A b.f. is said to be consonant if its focal elements are nested:* $E_1 \subset \cdots \subset E_m$.

Proposition 1 [5] illustrates some of their properties.

Proposition 1 *If b is a belief function with plausibility function pl_b, then the following conditions are equivalent:*

(1) *b is consonant;*
(2) $b(A \cap B) = \min(b(A), b(B))$ *for every* $A, B \subseteq \Theta$*;*
(3) $pl_b(A \cup B) = \max(pl_b(A), pl_b(B))$ *for every* $A, B \subseteq \Theta$*;*
(4) $pl_b(A) = \max_{x\in A} pl_b(x)$ *for all non-empty* $A \subseteq \Theta$*.*

Possibility theory [9] is based on a different description of uncertainty called *possibility measure.*

Definition 2 *A* possibility measure *on a domain* Θ *is a function* $Pos : 2^\Theta \rightarrow [0, 1]$ *such that* $Pos(\emptyset) = 0$, $Pos(\Theta) = 1$ *and* $Pos(\bigcup_i A_i) = \sup_i Pos(A_i)$ *for any family* $\{A_i | A_i \in 2^\Theta, i \in I\}$ *where* I *is an arbitrary set index.*

Each possibility measure is uniquely characterized by a *membership function* or *possibility distribution* $\pi : \Theta \rightarrow [0, 1]$ s.t. $\pi(x) \doteq Pos(\{x\})$ via the formula $Pos(A) = \sup_{x \in A} \pi(x)$. $Nec(A) = 1 - Pos(A^c)$ is called *necessity measure.* Many studies have pointed out that necessity measures coincide in the theory of evidence with the class of consonant belief functions. Let us call *plausibility assignment* (pl.ass.) $\bar{pl}_b$ [28] the restriction of the plausibility function to singletons $\bar{pl}_b(x) = pl_b(\{x\})$. From Condition 4 of Proposition 1 it follows immediately that

Proposition 2 *The plausibility function* pl_b *associated with a belief function* b *on a domain* Θ *is a possibility measure iff* b *is consonant, and in this case the membership function coincides with the plausibility assignment:* $\pi = \bar{pl}_b$. *Equivalently, a b.f.* b *is a necessity measure iff* b *is consonant.*

Possibility theory (in the finite case) is then embedded in the ToE. Studying the geometry of consonant belief functions is then equivalent to extending the geometric approach to the theory of evidence to possibility theory, in a step towards a unified geometric approach to uncertainty.

3 A geometric approach to the theory of evidence

Given a frame Θ, each belief function $b : 2^\Theta \rightarrow [0, 1]$ is completely specified by its $N - 2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$, $N \doteq 2^n$ ($n \doteq |\Theta|$), (as $b(\emptyset) = 0$, $b(\Theta) = 1$ for all b.f.s) and can then be represented as a point of $\mathbb{R}^{N-2}$.
We can introduce a linear order on the set of all subsets of $\Theta = \{x_1, ..., x_n\}$. For instance, given two subsets $A = \{x_{i_1}, ..., x_{i_l}\}$ and $B = \{x_{j_1}, ..., x_{j_m}\}$ we can say that $A < B$ if and only if either $l = |A| < m = |B|$ or $x_{i_k} < x_{j_k}$ with k the smallest index such that $x_{i_k} \neq x_{j_k}$.
We can then represent each b.f. b as the vector of $\mathbb{R}^{N-2}$

$$b = [b(A_1), ..., b(A_{N-2})]' \tag{3}$$

where $A_1 < ... < A_{N-2}$ is the linear order introduced above.
We call *belief space* $\mathcal{B}$ the set of points of $\mathbb{R}^{N-2}$ which correspond to a belief

function. Let us denote by

$$b_A \doteq b \in \mathcal{B} \; s.t. \; m_b(A) = 1, \; \forall B \subseteq \Theta \; s.t. \; B \neq A \; m_b(B) = 0 \tag{4}$$

the *categorical* [29] belief function assigning all the mass to a single subset $A \subseteq \Theta$. It can be proven that, denoting by $\mathcal{E}_b$ the list of focal elements of b,

Theorem 1 *The set of all the belief functions with focal elements in a given collection L is closed and convex in $\mathcal{B}$: $\{b : \mathcal{E}_b \subseteq L\} = Cl(b_A : A \in L)$, where Cl denotes the convex closure operator:*

$$Cl(b_1, ..., b_k) = \left\{ b \in \mathcal{B} : b = \alpha_1 b_1 + \cdots + \alpha_k b_k, \sum_i \alpha_i = 1, \; \alpha_i \geq 0 \; \forall i \right\}. \tag{5}$$

The following is then just a consequence of Theorem 1.

Corollary 1 *The belief space $\mathcal{B}$ coincides with the convex closure of all the categorical belief functions b_A: $\mathcal{B} = Cl(b_A, \emptyset \subsetneq A \subseteq \Theta)$.*

An n-dimensional *simplex* is the convex closure $Cl(x_1, ..., x_{n+1})$ of $n+1$ (affinely independent [1]) points $x_1, ..., x_{n+1}$ of the Euclidean space $\mathbb{R}^n$. The *faces* of an n-dimensional simplex are all the possible simplices generated by a subset of its vertices, i.e. $Cl(x_{j_1}, ..., x_{j_k})$ with $\{j_1, ..., j_k\} \subset \{1, ..., n+1\}$.
As it is not hard to see that the basis b.f.s b_A are affinely independent, Corollary 1 states that the belief space $\mathcal{B}$ is a simplex (Figure 1-left). Moreover, each belief function $b \in \mathcal{B}$ can be written as a convex sum as

$$b = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) b_A; \tag{6}$$

m_b is nothing but the set of simplicial coordinates of b in the simplex $\mathcal{B}$. As a probability is a belief function assigning non zero masses to singletons only, Theorem 1 implies that the set $\mathcal{P}$ of all Bayesian b.f.s is a part of the border of $\mathcal{B}$, precisely the simplex determined by all categorical b.f.s associated with singletons [2]: $\mathcal{P} = Cl(b_x, x \in \Theta)$.
Some other faces of the belief space also have an intuitive meaning in terms of belief. Consider the segments $Cl(b_\Theta, b_A)$ joining the vacuous belief function b_Θ ($m_{b_\Theta}(\Theta) = 1$, while $\forall B \subseteq \Theta$ s.t. $B \neq A$ we have that $m_{b_\Theta}(B) = 0$) with the categorical b.f. b_A (4). Points of $Cl(b_\Theta, b_A)$ can be written as a convex

[1] An *affine combination* of k points $v_1, ..., v_k \in \mathbb{R}^m$ is a sum $\alpha_1 v_1 + \cdots + \alpha_k v_k$ such that $\sum_i \alpha_i = 1$. The affine subspace generated by the points $v_1, ..., v_k \in \mathbb{R}^m$ is the set $\{v \in \mathbb{R}^m : v = \alpha_1 v_1 + \cdots + \alpha_k v_k, \sum_i \alpha_i = 1\}$. If $v_1, ..., v_k$ generate an affine space of dimension k they are said to be *affinely independent*.

[2] With a harmless abuse of notation we denote the categorical b.f. associated with a singleton x by b_x instead of $b_{\{x\}}$, and write $m_b(x), pl_b(x)$ instead of $m_b(\{x\}), pl_b(\{x\})$.

combination as $b = \alpha b_A + (1-\alpha) b_\Theta$. Since convex combinations as b.p.a.s in $\mathcal{B}$, such a b.f. b has b.p.a.

$$m_b(A) = \alpha, \;\; m_b(\Theta) = 1 - \alpha$$

i.e. b is a *simple support function* focused on A [5].
Accordingly the union of these segments for all events A, $\mathcal{S} = \bigcup_{A \subseteq \Theta} Cl(b_\Theta, b_A)$, is the region of simple support belief functions on Θ.

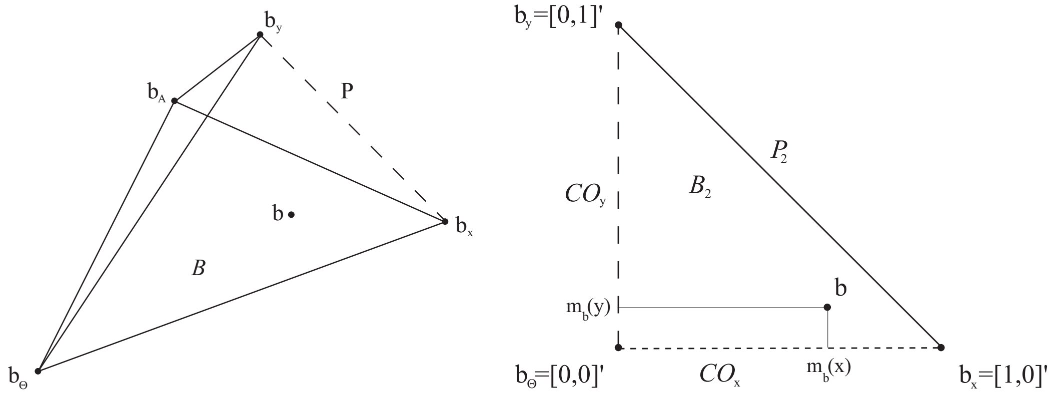

Fig. 1. Left: Simplicial structure of the belief space $\mathcal{B}$: Its vertices are all the categorical belief functions b_A represented as vectors of the Cartesian space $\mathbb{R}^{N-2}$. The region $\mathcal{P}$ of the Bayesian belief functions is one of its faces: $Cl(b_x, x \in \Theta)$. Right: The belief space $\mathcal{B}_2$ for a binary frame is a triangle in $\mathbb{R}^2$ whose vertices are the categorical belief functions focused on $\{x\}, \{y\}$ and Θ (b_x, b_y, b_Θ) respectively. The Bayesian region $\mathcal{P}_2$ is the segment $Cl(b_x, b_y)$, while consonant belief functions are constrained to belong to the union of the two segments $\mathcal{CO}_x = Cl(b_\Theta, b_x)$ and $\mathcal{CO}_y = Cl(b_\Theta, b_y)$.

3.1 Binary frame

As an example let us consider a frame of discernment containing only two elements, $\Theta_2 = \{x, y\}$. In this very simple case each b.f. $b : 2^{\Theta_2} \to [0,1]$ is completely determined by its belief values $b(x), b(y)$, as $b(\Theta) = 1$ and $b(\emptyset) = 0$ $\forall b$. We can then collect them in a vector of $\mathbb{R}^{N-2} = \mathbb{R}^2$ (since $N = 2^2 = 4$) according to the linear order (3) defined above:

$$[b(x) = m_b(x), b(y) = m_b(y)]' \in \mathbb{R}^2. \tag{7}$$

Since $m_b(x) \geq 0$, $m_b(y) \geq 0$, and $m_b(x) + m_b(y) \leq 1$ we can easily infer that the set $\mathcal{B}_2$ of all the possible belief functions on Θ_2 can be depicted as the triangle in the Cartesian plane of Figure 1-right, whose vertices are the points

$$b_\Theta = [0,0]', \; b_x = [1,0]', \; b_y = [0,1]'$$

which correspond (through Equation (7)) respectively to the vacuous belief function b_Θ ($m_{b_\Theta}(\Theta) = 1$), the Bayesian b.f. b_x with $m_{b_x}(x) = 1$, and the Bayesian b.f. b_y with $m_{b_y}(y) = 1$. The region $\mathcal{P}_2$ of all Bayesian b.f.s on Θ_2 is in this case the diagonal line segment $Cl(b_x, b_y)$. On the other side, simple support functions focused on $\{x\}$ lie on the horizontal segment $Cl(b_\Theta, b_x)$, while simple support b.f. focused on $\{y\}$ form the vertical segment $Cl(b_\Theta, b_y)$. On $\Theta_2 = \{x, y\}$ consonant belief functions can have as chain of focal elements either $\{\{x\}, \Theta_2\}$ or $\{\{y\}, \Theta_2\}$. Therefore all co.b.f.s on Θ_2 are simple support functions, and their region $\mathcal{CO}_2$ is the union of two segments (see Figure 1):

$$\mathcal{CO}_2 = \mathcal{S}_2 = \mathcal{CO}_x \cup \mathcal{CO}_y = Cl(b_\Theta, b_x) \cup Cl(b_\Theta, b_y).$$

In the rest of the paper we will study the geometry of consonant belief functions in the general case of arbitrary frames.

4 Consonant subspace

The geometric interpretation of belief functions puts indeed the results of Section 2.1 in a different light. Using the language of convex geometry we can pose the problem of finding the region of $\mathcal{B}$ whose points correspond to consonant belief functions.

4.1 Chains of subsets as consonant belief functions

Where arbitrary belief functions do not suffer from restrictions on their list of focal elements, consonant b.f.s are characterized by the fact that their focal elements can be rearranged into a totally ordered set by set inclusion.
The power set 2^Θ of a frame is a *partially ordered set* with respect to the set-theoretic inclusion. In other words, the relation $\subseteq$ possess three properties: reflexivity (whenever $A \subseteq \Theta$, $A \subseteq A$), antisymmetry ($A \subseteq B$ and $B \subseteq A$ implies $A = B$), and transitivity ($A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$). A *chain* of a poset is a collection of pairwise comparable elements (*totally ordered set*). The possible lists of focal elements associated with consonant belief functions then correspond to all the possible chains of subsets $A_1 \subseteq ... \subseteq A_m$ in the partially ordered set $(2^\Theta, \subseteq)$.
Now, Theorem 1 implies that the b.f.s whose focal elements belong to a chain $\mathcal{C} = \{A_1, \cdots, A_m\}$ form the simplex $Cl(b_{A_1}, ..., b_{A_m})$ (remember that the b_A's are affinely independent). No matter what the basic probability assignment is, all the $b \in Cl(b_{A_1}, ..., b_{A_m})$ are consonant belief functions.
Let us denote with $n \doteq |\Theta|$ the cardinality of the frame Θ. Since each chain in $(2^\Theta, \subseteq)$ is a subset of a maximal chain (a chain including subsets of any size

from 1 to n) the region of co.b.f.s turns out to be the union of a collection of simplices, each of them associated with a maximal chain $\mathcal{C}$:

$$\mathcal{CO} = \bigcup_{\mathcal{C}=A_1\subseteq\cdots\subseteq A_n} Cl(b_{A_1}, \cdots, b_{A_n}).$$

The number of maximal simplices of $\mathcal{CO}$ is then the number of maximal chains in $(2^\Theta, \subseteq)$, i.e.

$$\prod_{k=1}^{n} \binom{k}{1} = n!$$

since given a size k set we can build a new set containing it by just choosing one of the remaining elements. Since the length of a maximal chain is $|\Theta| = n$, the dimension of these convex components is $\dim Cl(b_{A_1}, ..., b_{A_n}) = n - 1$.
Each categorical belief function b_A obviously belongs to several distinct components. In particular, if $|A| = k$ the total number of maximal chains containing A is $(n-k)!k!$ since in the power set of A the number of maximal chains is $k!$, while to get a chain from A to Θ we just have to add an element of $A^c = \Theta \setminus A$ (whose size is $n-k$) at each step. $(n-k)!k!$ is then also the number of maximal simplices of $\mathcal{CO}$ containing b_A.
In particular, each vertex b_x of the probabilistic subspace $\mathcal{P}$ (for which $|\{x\}| = k = 1$) belongs to a sheaf of $(n-1)!$ convex components of the consonant subspace. Clearly the maximum number of simplices is $n!$, obtained for $k = n$ (the vacuous belief function b_Θ). An obvious remark is that $\mathcal{CO}$ *is connected*, for each convex component is obviously connected, and each pair of such components has at least b_Θ as intersection.

4.2 Ternary case

Let us consider, as an example, the case of a frame of size 3: $\Theta = \{x, y, z\}$. Belief functions $b \in \mathcal{B}_3$ can be written as 6-dimensional vectors according to the linear order (3)

$$[b(x), b(y), b(z), b(\{x, y\}), b(\{x, z\}), b(\{y, z\})]'.$$

All the possible maximal chains are in this case

$$\begin{array}{lll} \{x\} \subset \{x,z\} \subset \Theta & \{y\} \subset \{x,y\} \subset \Theta & \{z\} \subset \{y,z\} \subset \Theta \\ \{x\} \subset \{x,y\} \subset \Theta & \{y\} \subset \{y,z\} \subset \Theta & \{z\} \subset \{x,z\} \subset \Theta. \end{array}$$

Each singleton is then associated with 2 chains, and the total number of convex components, whose dimension is $|\Theta| - 1 = 2$, is $3! = 6$:

$$Cl(b_x, b_{\{x,z\}}, b_\Theta) \quad Cl(b_y, b_{\{x,y\}}, b_\Theta) \quad Cl(b_z, b_{\{y,z\}}, b_\Theta)$$
$$Cl(b_x, b_{\{x,y\}}, b_\Theta) \quad Cl(b_y, b_{\{y,z\}}, b_\Theta) \quad Cl(b_z, b_{\{x,z\}}, b_\Theta).$$

Each 2-dimensional simplex (for instance $Cl(b_x, b_{\{x,z\}}, b_\Theta)$) has an intersection of dimension $|\Theta| - 2 = 1$ ($Cl(b_{\{x,z\}}, b_\Theta)$) with a single other component ($Cl(b_z, b_{\{x,z\}}, b_\Theta)$) associated with a different element of Θ.
The geometry of the ternary frame can then be represented as in Figure 2-a, where the belief space is 6-dimensional $\mathcal{B}_3 = Cl(b_x, b_y, b_z, b_{\{x,y\}}, b_{\{x,z\}}, b_{\{y,z\}}, b_\Theta)$, its probabilistic face is a 2-dimensional simplex $\mathcal{P}_3 = Cl(b_x, b_y, b_z)$, and the consonant subspace $\mathcal{CO}_3$ is given by the union of the maximal simplices listed above.

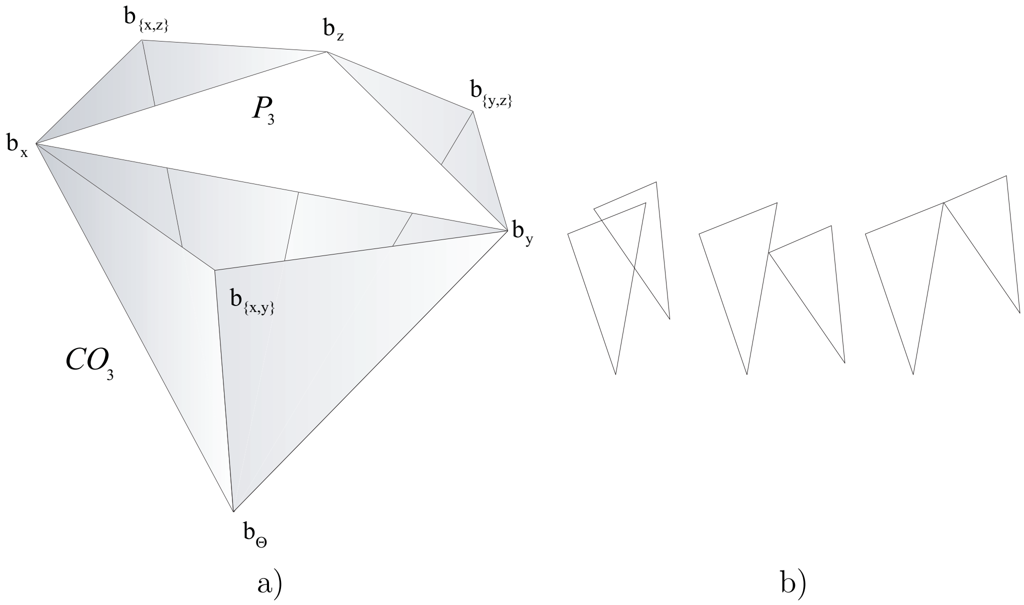

Fig. 2. a) The simplicial complex $\mathcal{CO}_3$ of all the consonant belief functions for a ternary frame Θ_3. The complex is composed by $n! = 3! = 6$ convex components of dimension $n - 1 = 2$, each vertex of $\mathcal{P}_3$ being shared by $(n-1)! = 2! = 2$ of them. The region is connected, and is part of the border $\partial\mathcal{B}_3$ of the belief space $\mathcal{B}_3$. b) Constraint on the intersection of simplices in a complex. Only the right-hand pair of triangles meets condition (2) of the definition of simplicial complex.

4.3 Consonant subspace as simplicial complex

These properties of $\mathcal{CO}$ can be summarized by means of another concept of convex geometry, which generalizes that of simplex [27].

Definition 3 *A* simplicial complex *is a collection* Σ *of simplices of arbitrary dimensions possessing the following properties:*

(1) if a simplex belongs to Σ, then all its faces of any dimension belong to Σ;
(2) the intersection of two simplices in the complex is a face of both.

Let us consider for instance two triangles on the plane (2-dimensional simplices). Roughly speaking, the second condition says that the intersection of those triangles cannot contain points of their interiors (Figure 2-b-right). It cannot be any subset of their borders either (middle), but has to be a face (right, in this case a single vertex). Note that if two simplices intersect in a face τ, they obviously intersect in every face of τ.

Theorem 2 *$\mathcal{CO}$ is a simplicial complex included in the belief space $\mathcal{B}$.*

Proof. Property (1) of Definition 3 is trivially satisfied. As a matter of fact, if a simplex $Cl(b_{A_1}, ..., b_{A_n})$ corresponds to a chain $A_1 \subseteq ... \subseteq A_n$ in the poset $(2^\Theta, \subseteq)$, clearly each face of this simplex correspond to a subchain in 2^Θ, and then to a simplex of consonant belief functions. About property (2), let us consider the intersection of two arbitrary simplices in the complex

$$Cl(b_{A_1}, ..., b_{A_{n_1}}) \cap Cl(b_{B_1}, ..., b_{B_{n_2}})$$

associated with the pair of chains $\mathcal{A} = \{A_1, ..., A_{n_1}\}$ and $\mathcal{B} = \{B_1, ..., B_{n_2}\}$ respectively. As the vectors $\{b_A, \emptyset \subsetneq A \subsetneq \Theta\}$ are linearly independent in $\mathbb{R}^{N-2}$, no linear combination of b_{B_i}'s can yield an element of $span(b_{A_1}, ..., b_{A_{n_1}})$, unless some of those vectors coincide. In this case the desired intersection is

$$Cl(b_{C_{i_1}}, ..., b_{C_{i_k}}) \qquad (8)$$

where

$$\mathcal{C} = \{C_{i_j}, j = 1, ..., k\} = \mathcal{A} \cap \mathcal{B}$$

with $k < n_1, n_2$. But then $\mathcal{C}$ is a subchain of both $\mathcal{A}$ and $\mathcal{B}$, so that (8) is a face of both $Cl(b_{A_1}, ..., b_{A_{n_1}})$ and $Cl(b_{B_1}, ..., b_{B_{n_2}})$. □

As Figure 2 shows, $\mathcal{P}$ and the maximal simplices of $\mathcal{CO}$ have the same dimension, and are both part of the boundary $\partial\mathcal{B}$ of the belief space.

5 Properties of the consonant subspace

It is worth to get a bit deeper in our understanding of the geometry of the consonant subspace.

5.1 Congruence of the convex components of $\mathcal{CO}$

As the binary case suggests, all maximal simplices of the consonant complex are congruent, i.e. they can be mapped onto each other by means of a rigid transformation. As we have seen in Section 3.1, the two components $\mathcal{CO}_x = Cl(b_\Theta, b_x)$ and $\mathcal{CO}_y = Cl(b_\Theta, b_y)$ are segments of the same size

$$\|\mathcal{CO}_x\| = \|b_x - b_\Theta\| = \|b_x\| = \|[1,0]'\| = 1 = \|b_y - b_\Theta\| = \|\mathcal{CO}_y\|.$$

We can get an intuition about how to prove this conjecture in the general case by studying the more significant ternary case. From Section 4.2,

$$\begin{array}{ccc} & & Cl(b_x, b_{\{x,y\}}, b_\Theta) \\ Cl(b_x, b_\Theta) & & \|b_x - b_\Theta\| = \|b_x\| = \|[1\ 0\ 0\ 1\ 1\ 0]'\| = \sqrt{3} \\ Cl(b_{\{x,y\}}, b_\Theta) & \leftrightarrow & \|b_{\{x,y\}} - b_\Theta\| = \|b_{\{x,y\}}\| = \|[0\ 0\ 0\ 1\ 0\ 0]'\| = 1 \\ Cl(b_x, b_{\{x,y\}}) & & \|b_x - b_{\{x,y\}}\| = \|[1\ 0\ 0\ 0\ 1\ 0]'\| = \sqrt{2} \end{array}$$

$$\begin{array}{ccc} & & Cl(b_x, b_{\{x,z\}}, b_\Theta) \\ Cl(b_x, b_\Theta) & & \|b_x - b_\Theta\| = \|b_x\| = \|[1\ 0\ 0\ 1\ 1\ 0]'\| = \sqrt{3} \\ Cl(b_{\{x,z\}}, b_\Theta) & \leftrightarrow & \|b_{\{x,z\}} - b_\Theta\| = \|b_{\{x,z\}}\| = \|[0\ 0\ 0\ 0\ 1\ 0]'\| = 1 \\ Cl(b_x, b_{\{x,z\}}) & & \|b_x - b_{\{x,z\}}\| = \|[1\ 0\ 0\ 1\ 0\ 0]'\| = \sqrt{2} \end{array}$$

$$\begin{array}{ccc} & & Cl(b_y, b_{\{x,y\}}, b_\Theta) \\ Cl(b_y, b_\Theta) & & \|b_y - b_\Theta\| = \|b_y\| = \|[0\ 1\ 0\ 1\ 0\ 1]'\| = \sqrt{3} \\ Cl(b_{\{x,y\}}, b_\Theta) & \leftrightarrow & \|b_{\{x,y\}} - b_\Theta\| = \|b_{\{x,y\}}\| = \|[0\ 0\ 0\ 1\ 0\ 0]'\| = 1 \\ Cl(b_y, b_{\{x,y\}}) & & \|b_y - b_{\{x,y\}}\| = \|[0\ 1\ 0\ 0\ 0\ 1]'\| = \sqrt{2} \end{array}$$

$$\begin{array}{ccc} & & Cl(b_z, b_{\{x,z\}}, b_\Theta) \\ Cl(b_z, b_\Theta) & & \|b_z - b_\Theta\| = \|b_z\| = \|[0\ 0\ 1\ 0\ 1\ 1]'\| = \sqrt{3} \\ Cl(b_{\{x,z\}}, b_\Theta) & \leftrightarrow & \|b_{\{x,z\}} - b_\Theta\| = \|b_{\{x,z\}}\| = \|[0\ 0\ 0\ 0\ 1\ 0]'\| = 1 \\ Cl(b_z, b_{\{x,z\}}) & & \|b_z - b_{\{x,z\}}\| = \|[0\ 0\ 1\ 0\ 0\ 1]'\| = \sqrt{2} \end{array}$$

the 1-dimensional faces of each pair of maximal simplices can be put into a 1-1 correspondence. For instance, considering the pair of triangles $Cl(b_x, b_{\{x,y\}}, b_\Theta)$, $Cl(b_z, b_{\{x,z\}}, b_\Theta)$, the desired correspondence is

$$Cl(b_x, b_\Theta) \leftrightarrow Cl(b_z, b_\Theta), \qquad Cl(b_{\{x,y\}}, b_\Theta) \leftrightarrow Cl(b_{\{x,z\}}, b_\Theta)$$
$$Cl(b_x, b_{\{x,y\}}) \leftrightarrow Cl(b_z, b_{\{x,z\}})$$

as those pairs of segments have the same norm.
We can prove that in the general case too.

Theorem 3 *All the maximal simplices of the consonant subspace are congruent.*

Proof. To get a proof for the general case we need to find a 1-1 map between 1-dimensional sides of two any maximal simplices. Let $\mathcal{A} = \{A_1 \subset \cdots \subset A_i \subset \cdots \subset A_n = \Theta\}$, $\mathcal{B} = \{B_1 \subset \cdots \subset B_i \subset \cdots \subset B_n = \Theta\}$ be the associated maximal chains. It is easy to see that we need to associate pairs of events with the same cardinality:

$$Cl(b_{A_i}, b_{A_j}) \leftrightarrow Cl(b_{B_i}, b_{B_j}), \quad |A_i| = |B_i| = i,\ |A_j| = |B_j| = j > i.$$

As a matter of fact, the categorical b.f. b_{A_i} is such that $b_{A_i}(B) = 1$ when $B \supseteq A_i$, $b_{A_i}(B) = 0$ otherwise. On the other side $b_{A_j}(B) = 1$ when $B \supseteq A_j \supset A_i$, $b_{A_j}(B) = 0$ otherwise, since $A_j \supset A_i$ by hypothesis. Hence

$$|b_{A_i} - b_{A_j}(B)| = 1 \Leftrightarrow B \supseteq A_i, B \not\supset A_j$$

so that

$$\|b_{A_i} - b_{A_j}\|_2 = \sqrt{|\{B \subseteq \Theta : B \supseteq A_i, B \not\supset A_j\}|} = \sqrt{|A_j \setminus A_i|}.$$

But this is true for each similar pair in any other maximal chain, so that

$$\|b_{A_i} - b_{A_j}\|_2 = \|b_{B_i} - b_{B_j}\|_2 \quad \forall i, j \in [1, ..., n]$$

for each pair of maximal simplices of $\mathcal{CO}$. Because of the generalization of a well known Euclid's theorem this implies that the two simplices are congruent [3]: $Cl(b_{A_1}, \cdots, b_\Theta) \sim Cl(b_{B_1}, \cdots, b_\Theta)$. □

It is easy to see that the components of $\mathcal{CO}$ are *not* congruent with $\mathcal{P}$, even though they have both dimension $n-1$. In the binary case, for instance,

$$\mathcal{P} = Cl(b_x, b_y), \quad \|\mathcal{P}\| = \|b_y - b_x\| = \sqrt{2}$$

while $\|\mathcal{CO}_x\| = \|\mathcal{CO}_y\| = 1$.

[3] Notice that, although this holds for *simplices* (generalized triangles), the same is not true for *polytopes* in general, i.e. convex closures of $k+1$ vertices which generates an affine space of dimension smaller than k. Think of a square and a rhombus with sides of length 1.

The analysis of the norm of the difference of two categorical belief functions tells us more about the nature of the maximal simplices of the consonant subspace. We know from [30] that in $\mathbb{R}^{N-2}$ each triangle

$$Cl(b_\Theta, b_B, b_A)$$

with $\Theta \supset B \supset A$ is a right triangle with right angle $\widehat{b_\Theta b_B b_A}$.
Indeed we ca prove a much more general result.

Theorem 4 *If* $A_i \supset A_j \supset A_k$ *then* $\widehat{b_{A_i} b_{A_j} b_{A_k}} = \pi/2$.

Proof. As $A_i \supset A_j \supset A_k$ we can write

$$b_{A_j} - b_{A_i}(B) = \begin{cases} 1 & B \supset A_j, B \not\supset A_i \\ 0 & otherwise \end{cases} \quad b_{A_k} - b_{A_i}(B) = \begin{cases} 1 & B \supset A_k, B \not\supset A_i \\ 0 & otherwise \end{cases}$$

$$b_{A_k} - b_{A_j}(B) = \begin{cases} 1 & B \supset A_k, B \not\supset A_j \\ 0 & otherwise \end{cases}$$

which implies

$$b_{A_i} - b_{A_j}(B) = 1 \Rightarrow B \not\supset A_i, B \supset A_j \Rightarrow b_{A_j} - b_{A_k}(B) = 0$$

and vice-versa, so that $\langle b_{A_i} - b_{A_j}, b_{A_j} - b_{A_k} \rangle = 0$ and $\widehat{b_{A_i} b_{A_j} b_{A_k}}$ is $\pi/2$. □

All triangles $Cl(b_{A_i}, b_{A_j}, b_{A_k})$ such that $A_i \supset A_j \supset A_k$ are right triangles. But as each simplicial component $\mathcal{CO}_\mathcal{C}$ of the consonant complex has vertices associated with the elements $A_1 \subseteq ... \subseteq A_n$ of a maximal chain, any three of them will also form a chain. Hence all 2-dimensional faces of all components of $\mathcal{CO}$ are right triangles: all its 3-dimensional faces (tetrahedrons) have as faces right triangles (Figure 3).

6 Geometry of outer consonant approximations in the consonant simplex

The geometric approach to the ToE was originally motivated by the search for Bayesian approximations of belief functions. We have just seen that it can nevertheless be extended to necessity measures, opening the way to a geometric analysis of the consonant approximation problem, i.e. the question

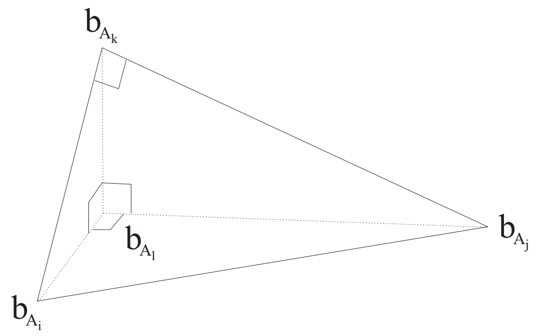

Fig. 3. All the tetrahedrons $Cl(b_{A_i}, b_{A_j}, b_{A_k}, b_{A_l})$ formed by vertices of a maximal simplex of the consonant subspace, $A_i \subset A_j \subset A_k \subset A_l$, have all right triangles as faces.

of finding the co.b.f. which minimizes a certain distance from the original b.f. b:

$$\hat{co} = \arg\min_{co \in \mathcal{CO}} dist(co, b). \tag{9}$$

As consonant belief functions represent necessity measures in the theory of belief functions, finding the "best" consonant approximation of a b.f. is equivalent to approximating a belief measure with a necessity measure.

6.1 Outer consonant approximations

Several partial orderings between belief functions have been introduced [31,32], in connection with the so-called "least commitment principle". The latter plays a similar role in the ToE as the principle of maximum entropy does in Bayesian theory. It postulates that, given a set of b.p.a.s compatible with a set of constraints, the most appropriate is the least informative (according to one of those orderings). In particular, b.f.s admit the following order relation

$$b \leq b' \equiv \forall A \subseteq \Theta \;\; b(A) \leq b'(A) \tag{10}$$

called *weak inclusion*. It is then possible to introduce the notion of *outer consonant approximations* [10] of a belief function b, i.e. those co.b.f.s such that $\forall A \subseteq \Theta$ $co(A) \leq b(A)$ (or equivalently $\forall A \subseteq \Theta$ $pl_{co}(A) \geq pl_b(A)$). In other words we seek co.b.f.s which are less informative than b in the sense specified above.
With the purpose of finding outer approximations which are *maximal* with respect to the weak inclusion relation (10) Dubois and Prade have introduced two different families of approximations.
A first group of is obtained by considering all permutations ρ of the elements $\{x_1, ..., x_n\}$ of the frame of discernment Θ: $\{x_{\rho(1)}, ..., x_{\rho(n)}\}$.

The following family of nested sets can be then built

$$\{S_1^\rho = \{x_{\rho(1)}\}, S_2^\rho = \{x_{\rho(1)}, x_{\rho(2)}\}, ..., S_n^\rho = \{x_{\rho(1)}, ..., x_{\rho(n)}\}\}$$

so that a new belief function co^ρ can be defined with b.p.a.

$$m_{co^\rho}(S_j^\rho) = \sum_{i:\min\{l:E_i \subseteq S_l^\rho\}=j} m_b(E_i). \quad (11)$$

Analogously an iterative procedure can be defined in which all permutations ρ of the focal elements $\{E_1, ..., E_k\}$ of b are considered $\{E_{\rho(1)}, ..., E_{\rho(k)}\}$ and the following family of sets is introduced

$$\{S_1^\rho = E_{\rho(1)}, S_2^\rho = E_{\rho(1)} \cup E_{\rho(2)}, ..., S_k^\rho = E_{\rho(1)} \cup \cdots \cup E_{\rho(k)}\}$$

so that a new belief function c_ρ can be defined with b.p.a.

$$m_{c_\rho}(S_j^\rho) = \sum_{i:\min\{l:E_i \subseteq S_l^\rho\}=j} m_b(E_i). \quad (12)$$

In general, approximations of the second family (12) are generated by the first family (11) too [10,13].

6.2 Geometry in the binary case

Let us discuss the geometry of the set $O[b]$ of all outer consonant approximations of a belief function b. We will first have a look at the situation in the case study of a binary frame, to later move to arbitrary frames of discernment.
The set $O[b]$ in the binary belief space $\mathcal{B}_2$ is depicted in Figure 4-left (dashed lines), as the intersection of the region of the points b' such that $\forall A \subseteq \Theta$ $b'(A) \leq b(A)$, and the complex $\mathcal{CO} = \mathcal{CO}_x \cup \mathcal{CO}_y$ of consonant b.f.s. Among them, the co.b.f.s generated by the 6 possible permutations of focal elements as in Equation (12) correspond to the points $c_{\rho_1}, ..., c_{\rho_6}$ in Figure 4, i.e. the orthogonal projections of b onto $\mathcal{CO}_x$, $\mathcal{CO}_y$ respectively, and the vacuous b.f. $b_\Theta = \mathbf{0}$.

Let us denote by $O_{\mathcal{C}}[b]$ the intersection of the set $O[b]$ of all outer consonant approximations with the component $\mathcal{CO}_{\mathcal{C}}$ of the consonant complex, with $\mathcal{C}$ a maximal chain of 2^Θ. We can notice a number of interesting facts.
For each maximal chain $\mathcal{C}$:

(1) $O_{\mathcal{C}}[b]$ is convex (in this case $\mathcal{C} = \{x, \Theta\}$ or $\{y, \Theta\}$);
(2) $O_{\mathcal{C}}[b]$ is in fact a *polytope*, i.e. the convex closure of a number of vertices: in particular a segment in the binary case ($O_{x,\Theta}[b]$ or $O_{y,\Theta}[b]$);

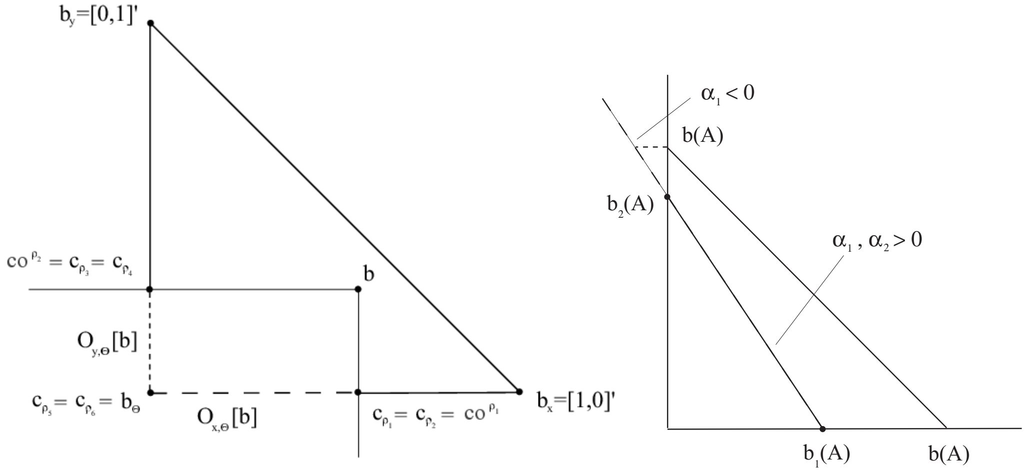

Fig. 4. Left: Geometry of outer consonant approximations of a belief function $b \in \mathcal{B}_2$. Right: The convex combination of two b.f.s weakly included in b is still weakly included in b: this does not hold for affine combinations.

(3) the maximal (with respect to (10)) outer approximation of b is one of the vertices of this polytope $O_{\mathcal{C}}[b]$, the one (co^{ρ}, Equation (11)) associated with the permutation ρ of singletons which generates the chain.

In the binary case there are just two such permutations, $\rho_1 = \{x, y\}$ and $\rho_2 = \{y, x\}$, which generate respectively the chains $\{x, \Theta\}$ and $\{y, \Theta\}$.
We will prove that all those properties indeed hold in the general case.

6.3 Convexity

Theorem 5 *Let b be a belief function on Θ. For each maximal chain $\mathcal{C}$ of 2^{Θ}, the set of outer consonant approximations $O_{\mathcal{C}}[b]$ of b which belong to the simplicial component $\mathcal{CO}_{\mathcal{C}}$ of the consonant space $\mathcal{CO}$ is convex.*

Proof. Consider two generic belief functions b_1, b_2 for which $\forall A \subseteq \Theta$ $b_1(A) \leq b(A)$, $b_2(A) \leq b(A)$. Then

$$\alpha_1 b_1(A) + \alpha_2 b_2(A) \leq \alpha_1 b(A) + \alpha_2 b(A) = (\alpha_1 + \alpha_2) b(A) = b(A) \qquad (13)$$

whenever $\alpha_1 + \alpha_2 = 1$, $\alpha_i \geq 0$. If α_1, α_2 are not guaranteed to be non-negative the sum $\alpha_1 b_1(A) + \alpha_2 b_2(A)$ can be greater than $b(A)$ (see Figure 4-right). Now, this holds in particular if the two b.f.s are consonant: Their convex combination, though, is obviously not guaranteed to be consonant.
If they both belong to the same maximal simplex of the consonant complex, however, their convex combination still lives in the simplex and $\alpha_1 b_1 + \alpha_2 b_2$ is both consonant and weakly included in b. □

6.4 Weak inclusion and mass re-assignment

A more cogent statement on the shape of $O[b]$ can be proven by means of the following result on the basic probability assignment of consonant belief functions weakly included in b.

Lemma 1 *Consider a belief function b with basic probability assignment m_b. A consonant belief function co is weakly included in b, for all $A \subseteq \Theta$ $co(A) \leq b(A)$, if and only if there is a choice of coefficients $\{\alpha_A^B, B \subseteq \Theta, A \supseteq B\}$ with*

$$\forall B \subseteq \Theta, \forall A \supseteq B, 0 \leq \alpha_A^B \leq 1 \qquad \forall B \subseteq \Theta, \sum_{A \supseteq B} \alpha_A^B = 1 \tag{14}$$

such that co has basic probability assignment

$$m_{co}(A) = \sum_{B \subseteq A} \alpha_A^B m_b(B). \tag{15}$$

Lemma 1 states that the b.p.a. of any outer consonant approximation of b is obtained by re-assigning the mass of each f.e. A of b to some $B \supseteq A$. We will extensively use this result in the following.

6.5 The polytopes $O_{\mathcal{C}}[b]$ of outer approximations

Summarizing, given a consonant belief function co weakly included in b, its focal elements will form a chain $\mathcal{C} = \{B_1, ..., B_n\}$ ($|B_i| = i$) associated with a specific maximal simplex of $\mathcal{CO}$. According to Lemma 1 the mass of each focal element A of b can be re-assigned to some of the events of the chain $B_1, ..., B_n$ which contain A in order to obtain co.
It is therefore natural to conjecture that, for each maximal simplex $\mathcal{CO}_{\mathcal{C}}$ of $\mathcal{CO}$ associated with a maximal chain $\mathcal{C}$, $O_{\mathcal{C}}[b]$ is the convex closure of the co.b.f.s $o^{\vec{B}}[b]$ with b.p.a.

$$m_{o^{\vec{B}}[b]}(B_i) = \sum_{A \subseteq \Theta : \vec{B}(A) = B_i} m_b(A) \tag{16}$$

each of them associated with an "assignment function"

$$\begin{aligned} \vec{B} &: 2^{\Theta} \rightarrow \mathcal{C} \\ A &\mapsto \vec{B}(A) \supseteq A \end{aligned} \tag{17}$$

which maps each event A to one of the events of the chain $\mathcal{C} = \{B_1 \subset ... \subset B_n\}$ which contains it. As a matter of fact,

Theorem 6 *For each simplicial component $\mathcal{CO}_\mathcal{C}$ of the consonant space associated with any maximal chain of focal elements $\mathcal{C} = \{B_1, ..., B_n\}$ the set of outer consonant approximation of any b.f. b is the convex closure*

$$O_\mathcal{C}[b] = Cl(o^{\vec{B}}[b], \forall \vec{B})$$

of the co.b.f.s (16) indexed by all admissible assignment functions (17).

In other words, $O_\mathcal{C}[b]$ is a *polytope*, the convex closure of a number of b.f.s whose number is equal to the number of assignment functions (17). Each $\vec{B}$ is characterized by assigning each event A to an element $B_i \supseteq A$ of the chain $\mathcal{C}$.

As we will see in the ternary example of Section 6.7 the points (16) are not guaranteed to be proper vertices of the polytope $O_\mathcal{C}[b]$. Some of them can be obtained as a convex combination of the others, i.e. they may lie on a side of the polytope.

6.6 Maximal outer approximations

6.6.1 Permutations of singletons and maximal approximations

We can prove instead that the outer approximation (11) obtained by permuting the singletons of Θ as in Section 6.1 is not only a pseudo-vertex of $O_\mathcal{C}[b]$, but it is an actual vertex, i.e. it *cannot* be obtained as a convex combination of the others. More precisely, all possible permutations of elements of Θ generate exactly $n!$ different outer approximations of b, each of which lies on a single simplicial component of the consonant complex. Each such permutation ρ generates a maximal chain $\mathcal{C}_\rho = \{S_1^\rho, ..., S_n^\rho\}$ of focal elements so that the corresponding b.f. will lie on $\mathcal{CO}_{\mathcal{C}_\rho}$.

Theorem 7 *The outer consonant approximation co^ρ (11) generated by a permutation ρ of the singletons is a vertex of $O_{\mathcal{C}_\rho}[b]$.*

In Section 6.2 we conjectured that, for each maximal chain $\mathcal{C}$, the maximal (with respect to the weak inclusion relation) outer consonant approximation of b is indeed one of the vertices of the simplex $O_\mathcal{C}[b]$. We can easily prove that the maximal outer approximation is indeed the vertex co^ρ associated with the corresponding permutation ρ of the singletons which generates the maximal chain $\mathcal{C} = \mathcal{C}_\rho$.
As a matter of fact by definition (11) co^ρ assigns the mass $m_b(A)$ of each focal element A to the smallest element of the chain containing A. By Lemma 1 each outer consonant approximation of b with chain $\mathcal{C}$, $co \in O_{\mathcal{C}_\rho}[b]$, is the result of re-distributing the mass of each focal element A to all its supersets in the chain $\{B_i \supseteq A, B_i \in \mathcal{C}\}$.

But then each such co is weakly included in co^ρ for its b.p.a. can be obtained by re-distributing the mass of the minimal superset B_j, where $j = \min\{i : B_i \subseteq A\}$, to all supersets of A.

Corollary 2 *The maximal outer consonant approximation with maximal chain $\mathcal{C}$ of a belief function b is the vertex (11) of $O_{\mathcal{C}_\rho}[b]$ associated with the permutation ρ of the singletons which generates $\mathcal{C} = \mathcal{C}_\rho$.*

6.6.2 Maximal approximations and lower chain measures

A different perspective on maximal outer consonant approximations is given by the notion of *chain measure* [33].
Let us $\mathcal{S}$ be a family of subsets of a non-empty set Θ containing $\emptyset$ and Θ itself. The "inner extension" of a monotone set function $\mu : \mathcal{S} \to [0, 1]$ (s.t. $A \subseteq B$ implies $\mu(A) \leq \mu(B)$) is $\mu_*(A) = \sup_{B\in\mathcal{S},B\subset A} \mu(B)$ (dually for the outer extension).

Definition 4 *A monotone set function $\beta : \mathcal{S} \to [0, 1]$ is called* lower chain measure*, if there is a chain w.r.t. set inclusion $\mathcal{C} \subset \mathcal{S}$ with $\emptyset, \Theta \in \mathcal{C}$ such that $\beta = (\beta|\mathcal{C})_*|\mathcal{S}$, i.e., β coincides with the inner extension of its restriction to the elements of the chain.*

We can prove that for a lower chain measure β on $\mathcal{S}$: $\beta(\cap_{A\in\mathcal{A}}) = \inf_{A\in\mathcal{A}} \beta(A)$ for all finite set systems $\mathcal{A}$ such that $\cap_{A\in\mathcal{A}} A \in \mathcal{S}$. If this property holds for *arbitrary* $\mathcal{A}$ and $\mathcal{S}$ is closed under arbitrary intersection, then β is called a necessity measure. Any necessity measure is a lower chain measure, but the converse does not hold. However, the class of necessity measures coincides with the class of lower chain measures if Θ is finite.
As consonant belief functions are necessity measures on finite domains, they are trivially also lower chain measures and vice-versa.

Now, let b be a belief function and $\mathcal{C}$ a maximal chain in 2^Θ. Then we can build a chain measure (consonant b.f.) associated with b as

$$b_\mathcal{C}(A) = \max_{B\in\mathcal{C},B\subseteq A} b(B). \qquad (18)$$

We can easily prove the following result.

Theorem 8 *The chain measure (18) associated with the maximal chain $\mathcal{C}$ coincides with the vertex co_ρ (11) of the polytope of outer consonant approximations $O_{\mathcal{C}_\rho}[b]$ of b associated with the permutation ρ of the elements of Θ which generates $\mathcal{C} = \mathcal{C}_\rho$.*

Proof. Let us denote as usual by $\{B_1, ..., B_n\}$ the elements of the maximal chain $\mathcal{C}$. By definition the masses co_ρ assigns to the elements of the chain are

$$m_{co_\rho}(B_i) = \sum_{B \subseteq B_i, B \not\subseteq B_{i-1}} m_b(B)$$

so that the belief value of co_ρ on an arbitrary event $A \subseteq \Theta$ can be written as

$$co_\rho(A) = \sum_{B_i \subseteq A, B_i \in \mathcal{C}} m_{co_\rho}(B_i) = \sum_{B_i \subseteq A} \sum_{B \subseteq B_i, B \not\subseteq B_{i-1}} m_b(B)$$
$$= \sum_{B \subseteq B_{i_A}} m_b(B) = b(B_{i_A})$$

where B_{i_A} is the largest element of the chain included in A. But then clearly, as the elements $B_1 \subset \cdots \subset B_n$ of the chain are obviously nested and any belief function b is monotone,

$$co_\rho(A) = b(B_{i_A}) = \max_{B_i \in \mathcal{C}, B_i \subseteq A} b(B_i)$$

i.e., co_ρ is indeed (18). □

The chain measure associated with a b.f. b and a maximal chain $\mathcal{C}$ *is* the maximal outer consonant approximation of b.

6.7 Example

To better understand the properties we just proved, let us consider as an example a belief function b on a ternary frame $\Theta = \{x, y, z\}$ and study the polytope of outer consonant approximations with focal elements $\mathcal{C} = \{\{x\}, \{x, y\}, \{x, y, z\}\}$. According to Theorem 6 this is the convex closure of all assignment functions $\vec{B} : 2^\Theta \to \mathcal{C}$: there are $\prod_{k=1}^{3} k^{2^{3-k}} = 1^4 \cdot 2^2 \cdot 3^1 = 12$ such functions. We list them here as vectors of the form $\vec{B} = [\vec{B}(\{x\}), \vec{B}(\{y\}), \vec{B}(\{z\}), \vec{B}(\{x, y\}), \vec{B}(\{x, z\}), \vec{B}(\{y, z\}), \vec{B}(\{x, y, z\})]'$, i.e.,

$\vec{B}_1 = \{x\}, \{x, y\}, \Theta, \{x, y\}, \Theta, \Theta, \Theta;$ $\vec{B}_7 = \{x, y\}, \Theta, \Theta, \{x, y\}, \Theta, \Theta, \Theta;$
$\vec{B}_2 = \{x\}, \{x, y\}, \Theta, \Theta, \Theta, \Theta, \Theta;$ $\vec{B}_8 = \{x, y\}, \Theta, \Theta, \Theta, \Theta, \Theta, \Theta;$
$\vec{B}_3 = \{x\}, \Theta, \Theta, \{x, y\}, \Theta, \Theta, \Theta;$ $\vec{B}_9 = \Theta, \{x, y\}, \Theta, \{x, y\}, \Theta, \Theta, \Theta;$
$\vec{B}_4 = \{x\}, \Theta, \Theta, \Theta, \Theta, \Theta, \Theta;$ $\vec{B}_{10} = \Theta, \{x, y\}, \Theta, \Theta, \Theta, \Theta, \Theta;$
$\vec{B}_5 = \{x, y\}, \{x, y\}, \Theta, \{x, y\}, \Theta, \Theta, \Theta;$ $\vec{B}_{11} = \Theta, \Theta, \Theta, \{x, y\}, \Theta, \Theta, \Theta;$
$\vec{B}_6 = \{x, y\}, \{x, y\}, \Theta, \Theta, \Theta, \Theta, \Theta;$ $\vec{B}_{12} = \Theta, \Theta, \Theta, \Theta, \Theta, \Theta, \Theta.$

They correspond to the following co.b.f.s with b.p.a. $[m(\{x\}), m(\{x,y\}), m(\Theta)]'$:

$$\begin{array}{llll}
o^{\vec{B}_1} = [m_b(x), & m_b(y) + m_b(x,y), & 1 - b(x,y) &]'; \\
o^{\vec{B}_2} = [m_b(x), & m_b(y), & 1 - m_b(x) - m_b(y) &]'; \\
o^{\vec{B}_3} = [m_b(x), & m_b(x,y), & 1 - m_b(x) - m_b(x,y) &]'; \\
o^{\vec{B}_4} = [m_b(x), & 0, & 1 - m_b(x) &]'; \\
o^{\vec{B}_5} = [0, & b(x,y), & 1 - b(x,y) &]'; \\
o^{\vec{B}_6} = [0, & m_b(x) + m_b(y), & 1 - m_b(x) - m_b(y) &]'; \\
o^{\vec{B}_7} = [0, & m_b(x) + m_b(x,y), & 1 - m_b(x) - m_b(x,y) &]'; \\
o^{\vec{B}_8} = [0, & m_b(x), & 1 - m_b(x) &]'; \\
o^{\vec{B}_9} = [0, & m_b(y) + m_b(x,y), & 1 - m_b(y) - m_b(x,y) &]'; \\
o^{\vec{B}_{10}} = [0, & m_b(y), & 1 - m_b(y) &]'; \\
o^{\vec{B}_{11}} = [0, & m_b(x,y), & 1 - m_b(x,y) &]'; \\
o^{\vec{B}_{12}} = [0, & 0, & 1 &]'.
\end{array} \quad (19)$$

Figure 5-left shows the resulting polytope $O_{\mathcal{C}}[b]$ for a belief function $m_b(x) = 0.3$, $m_b(y) = 0.5$, $m_b(\{x,y\}) = 0.1$, $m_b(\Theta) = 0.1$, in the component $\mathcal{CO}_{\mathcal{C}} = Cl(b_x, b_{\{x,y\}}, b_\Theta)$ of the consonant complex (black triangle in the figure). The polytope $O_{\mathcal{C}}[b]$ is plotted in red, together with all the 12 points (19) (red squares). Many of them lie on a side of the polytope. However, the point obtained by permutation of singletons (11) is an actual vertex (red star): it is the first $o^{\vec{B}_1}$ of the list (19).

It is interesting to point out how the points (19) are ordered with respect to the weak inclusion relation (we just need to apply its definition, or the re-distribution property of Lemma 1). The result is summarized in the graph of Figure 6. We can appreciate that the vertex $o^{\vec{B}_1}$ generated by singleton permutation is indeed the maximal outer approximation of b, as stated by Corollary 2.

7 Conclusions and perspectives

In this paper we extended the geometric approach to the ToE to consonant belief functions, proving that the region of all co.b.f.s in the belief space $\mathcal{B}$ has the form of a simplicial complex $\mathcal{CO}$, in which each convex component is a simplex associated with a maximal chain of events of the frame Θ. We proved that all the convex components of $\mathcal{CO}$ are congruent, and described

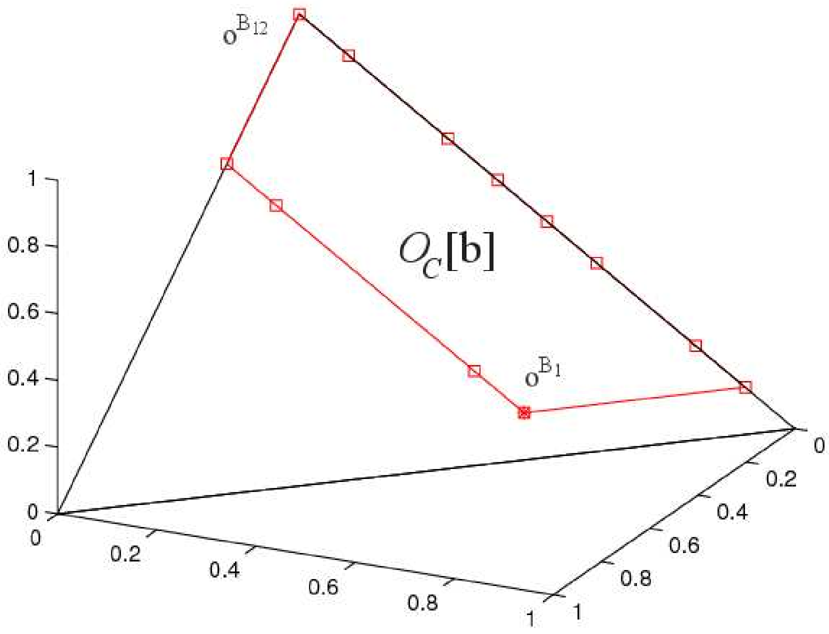

Fig. 5. Not all the points (16) associated with assignment functions are actual vertices of $O_{\mathcal{C}}[b]$. Here the polytope $O_{\mathcal{C}}[b]$ of outer consonant approximations for the belief function $m_b(x) = 0.3$, $m_b(y) = 0.5$, $m_b(\{x, y\}) = 0.1$, $m_b(\Theta) = 0.1$ defined on $\Theta = \{x, y, z\}$, with $\mathcal{C} = \{\{x\}, \{x, y\}, \Theta\}$ is plotted in red, together with all the 12 points (19) (red squares). Many of them lie on a side of the polytope. However, the point obtained by permutation of singletons (11) is an actual vertex (red star). The minimal and maximal outer approximations with respect to weak inclusion are $o^{\vec{B}_{12}}$ and $o^{\vec{B}_1}$, respectively.

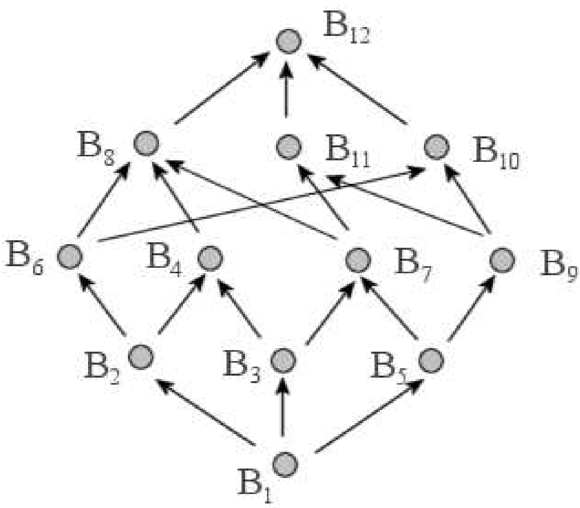

Fig. 6. Partial order of the points (19) with respect to the weak inclusion relation. For sake of simplicity we denote by B_i the co.b.f. $o^{\vec{B}_i}$ associated with the assignment function $\vec{B}_i$. An arrow from $\vec{B}_i$ to $\vec{B}_j$ stands for $o^{\vec{B}_j} \leq o^{\vec{B}_i}$.

in more detail their shape by showing that they can in fact be decomposed into rectangular triangles. We applied the formalism of simplicial complexes to the analysis of the consonant approximation problem, focusing in particular on the properties of outer consonant approximations. We showed that such approximations form a polytope in each maximal simplex of $\mathcal{CO}$. Finally, for a given chain the maximal outer approximation is a vertex of the corresponding polytope generated by a permutation of the elements of the frame, and coincides with the lower chain measure associated with the original belief function. As consonant belief functions correspond to finite necessity measures this work can be seen as a first step towards a unified geometric picture of all finite fuzzy measures.

Appendix

Proof of Lemma 1

(1) Sufficiency.

If Equation (15) holds for all focal elements $A \subseteq \Theta$ then

$$co(A) = \sum_{X \subseteq A} m_{co}(X) = \sum_{X \subseteq A} \sum_{B \subseteq X} \alpha_X^B m_b(B) = \sum_{B \subseteq A} m_b(B) \sum_{B \subseteq X \subseteq A} \alpha_X^B$$

where

$$\sum_{B \subseteq X \subseteq A} \alpha_X^B \leq \sum_{X \supseteq B} \alpha_X^B = 1$$

by Condition (14), so that

$$co(A) \leq \sum_{B \subseteq A} m_b(B) = b(A),$$

i.e., co is weakly included in b.

(2) Necessity.

Let us denote by $\mathcal{C} = \{B_1, ..., B_n\}$, $n = |\Theta|$ the chain of focal elements of co, and consider first the subsets $A \subseteq \Theta$ such that $A \not\supset B_1$ ($A \notin \mathcal{C}$). In this case

$$co(A) = 0 \leq m_b(A)$$

whatever the mass assignment of b. We then just need to focus on the elements $A = B_i \in \mathcal{C}$ of the chain.
We need to prove that for all $B_i \in \mathcal{C}$:

$$m_{co}(B_i) = \sum_{B \subseteq B_i} \alpha_{B_i}^B m_b(B) \qquad \forall i = 1, \cdots, n. \tag{20}$$

Let us use the notation $\alpha_i^B \doteq \alpha_{B_i}^B$ for sake of simplicity.
For each i we can sum the first i equations of the system (20) and obtain the equivalent system of equations:

$$co(B_i) = \sum_{B \subseteq B_i} \beta_i^B m_b(B) \qquad \forall i = 1, \cdots, n \tag{21}$$

as $co(B_i) = \sum_{j=1}^{i} m_{co}(B_j)$ for co is consonant. For all $B \subseteq \Theta$ the coefficients

$$\beta_i^B \doteq \sum_{j=1}^{i} \alpha_j^B$$

has to be such that

$$0 \leq \beta_{i_{min}}^B \leq \cdots \leq \beta_n^B = 1 \tag{22}$$

where $i_{min} = \min\{j : B_j \supseteq B\}$.

We can prove by induction on i that if co is weakly included in b, i.e., $co(B_i) \leq b(B_i)$ for all $i = 1, ..., n$, there exists a solution $\{\beta_i^B, B \subseteq B_i, i = 1, ..., n\}$ of the system (21) which meets the constraint (22).
Let us look for solutions of the form:

$$\left\{ \beta_i^B, B \subseteq B_{i-1}; \beta_i^B = \frac{co(B_i) - \sum_{X \subseteq B_{i-1}} \beta_i^X m_b(X)}{\sum_{X \subseteq B_i, X \not\subseteq B_{i-1}} m_b(X)}, B \subseteq B_i, B \not\subset B_{i-1} \right\} \tag{23}$$

in which the coefficients (variables) β_i^B associated with subsets of the previous f.e. B_{i-1} are left free, while *all* the coefficients associated with subsets that are in B_i but not in B_{i-1} are set to a *common* value, which depends on the free variables β_i^X, $X \subseteq B_{i-1}$.

Step $i = 1$. We get

$$\beta_1^{B_1} = \frac{co(B_1)}{m_b(B_1)}$$

which is such that $0 \leq \beta_1^{B_1} \leq 1$ as $co(B_1) \leq m_b(B_1)$, and trivially satisfies the first equation of system (21): $co(B_1) = \beta_1^{B_1} m_b(B_1)$.

Step i. We suppose there exists a solution (23) for $\{B \subseteq B_j, j = 1, ..., i-1\}$. First have to show that all solutions of the form (23) for i solve the i-th equation of system (21).
When we replace (23) in the i-th equation of the system we get (as the variables β_i^B in (23) do not depend on B for all $B \subseteq B_i, B \not\subset B_{i-1}$)

$$co(B_i) = \sum_{B \subseteq B_{i-1}} \beta_i^B m_b(B) + \left(\frac{co(B_i) - \sum_{X \subseteq B_{i-1}} \beta_i^X m_b(X)}{\sum_{X \subseteq B_i, X \not\subset B_{i-1}} m_b(X)} \right) \sum_{B \subseteq B_i, B \not\subset B_{i-1}} m_b(B),$$

i.e., $co(B_i) = co(B_i)$.

We also need to show, though, that there exist solutions of the above form (23) that meet the ordering constraint (22), i.e.,

$$0 \leq \beta_i^B \leq 1, \quad B \subseteq B_i, B \not\subset B_{i-1}; \qquad \beta_{i-1}^B \leq \beta_i^B \leq 1, \quad B \subseteq B_{i-1}. \tag{24}$$

The constraints (24) generate constraints on the free variables in (23), i.e., $\{\beta_i^B, B \subseteq B_{i-1}\}$. Given the shape of (23) those conditions (in the same order as in (24)) assume the form

$$\begin{cases} \sum_{B \subseteq B_{i-1}} \beta_i^B m_b(B) \leq co(B_i) \\ \sum_{B \subseteq B_{i-1}} \beta_i^B m_b(B) \geq co(B_i) - \sum_{B \subseteq B_i, B \not\subset B_{i-1}} m_b(B) \\ \beta_i^B \geq \beta_{i-1}^B \quad \forall B \subseteq B_{i-1} \\ \beta_i^B \leq 1 \quad \forall B \subseteq B_{i-1}. \end{cases} \tag{25}$$

Let us call 1.,2.,3.,4. the above constraints on the free variables $\{\beta_i^B, B \subseteq B_{i-1}\}$.

- 1. and 2. are trivially compatible;
- 1. is compatible with 3. as replacing $\beta_i^B = \beta_{i-1}^B$ into 1. yields (due to the $i-1$-th equation of the system)

$$\sum_{B \subseteq B_{i-1}} \beta_i^B m_b(B) = \sum_{B \subseteq B_{i-1}} \beta_{i-1}^B m_b(B) = co(B_{i-1}) \leq co(B_i);$$

- 4. is compatible with 2. as replacing $\beta_i^B = 1$ into 2. yields

$$\sum_{B \subseteq B_{i-1}} \beta_i^B m_b(B) = \sum_{B \subseteq B_{i-1}} m_b(B) \geq co(B_i) - \sum_{B \subseteq B_i, B \not\subset B_{i-1}} m_b(B)$$

  which is equivalent to

$$\sum_{B \subseteq B_i} m_b(B) = b(B_i) \leq co(B_i)$$

  which is true by hypothesis;
- 4. and 1. are clearly compatible, as we just need to choose β_i^B small enough;
- 2. and 3. are compatible, as we just need to choose β_i^B large enough.

In conclusion, all the constraints in Equation (25) are mutually compatible. Hence there exists an admissible solution to the i-th equation of system (21), which proves the induction step.

Proof of Theorem 6

We need to prove that:

(1) each co.b.f. $co \in \mathcal{CO}_\mathcal{C}$ such that $co(A) \leq b(A)$ for all $A \subseteq \Theta$ can be written as a convex combination of the points (16): $co = \sum_{\vec{B}} \alpha_{\vec{B}} o^{\vec{B}}[b]$, $\sum_{\vec{B}} \alpha_{\vec{B}} = 1$, $\alpha_{\vec{B}} \geq 0 \ \forall \vec{B}$;
(2) vice-versa, each convex combination of the $o^{\vec{B}}[b]$ satisfies $\sum_{\vec{B}} \alpha_{\vec{B}} o^{\vec{B}}[b](A) \leq b(A)$ for all $A \subseteq \Theta$.

Let us consider (2) first. By definition of b.f. $o^{\vec{B}}[b](A) = \sum_{B \subseteq A, B \in \mathcal{C}} m_{o^{\vec{B}}[b]}(B)$ where $m_{o^{\vec{B}}[b]}(B) = \sum_{X \subseteq B : \vec{B}(X) = B} m_b(X)$ so that

$$o^{\vec{B}}[b](A) = \sum_{B \subseteq A, B \in \mathcal{C}} \ \sum_{X \subseteq B : \vec{B}(X) = B} m_b(X) = \sum_{X \subseteq B_i : \vec{B}(X) = B_j, j \leq i} m_b(X) \qquad (26)$$

where B_i is the largest element of the chain $\mathcal{C}$ included in A. As $B_i \subseteq A$ (26) is obviously not larger than $\sum_{B \subseteq A} m_b(B) = b(A)$, so that $o^{\vec{B}}[b](A) \leq b(A)$ for all A. Hence $\forall A \subseteq \Theta$

$$\sum_{\vec{B}} \alpha_{\vec{B}} o^{\vec{B}}[b](A) \leq \sum_{\vec{B}} \alpha_{\vec{B}} b(A) = b(A) \sum_{\vec{B}} \alpha_{\vec{B}} = b(A).$$

Let us prove point (1). According to Lemma 1, if $\forall A \subseteq \Theta$ $co(A) \leq b(A)$ then the mass $m_{co}(B_i)$ of each event B_i of the chain is

$$m_{co}(B_i) = \sum_{A \subseteq B_i} m_b(A) \alpha^A_{B_i}. \qquad (27)$$

To prove (1) we then need to write (27) as a convex combination of the $m_{o^{\vec{B}}[b]}(B_i)$, i.e.

$$\sum_{\vec{B}} \alpha_{\vec{B}} o^{\vec{B}}[b](B_i) = \sum_{\vec{B}} \alpha_{\vec{B}} \sum_{X \subseteq B_i : \vec{B}(X) = B_i} m_b(X) = \sum_{X \subseteq B_i} m_b(X) \sum_{\vec{B}(X) = B_i} \alpha_{\vec{B}}.$$

In other words we need to show that the system of equations

$$\begin{cases} \alpha^A_{B_i} = \displaystyle\sum_{\vec{B}(A) = B_i} \alpha_{\vec{B}} \qquad \forall i = 1, ..., n; \quad \forall A \subseteq B_i \end{cases} \qquad (28)$$

has at least one solution $\{\alpha_{\vec{B}}\}$ such that $\sum_{\vec{B}} \alpha_{\vec{B}} = 1$ and $\forall \vec{B}$ $\alpha_{\vec{B}} \geq 0$. The normalization constraint is in fact trivially satisfied as from (28) it follows that

$$\sum_{B_i \supseteq A} \alpha^A_{B_i} = 1 = \sum_{B_i \supseteq A} \sum_{\vec{B}(A) = B_i} \alpha_{\vec{B}} = \sum_{\vec{B}} \alpha_{\vec{B}}$$

i.e. $\sum_{\vec{B}} \alpha_{\vec{B}} = 1$. Using the normalization constraint the system of equations (28) reduces to

$$\left\{ \alpha^A_{B_i} = \sum_{\vec{B}(A)=B_i} \alpha_{\vec{B}} \qquad \forall i = 1, ..., n-1; \quad \forall A \subseteq B_i. \right. \tag{29}$$

We can show that each equation in the reduced system (29) involves at least one variable $\alpha_{\vec{B}}$ which is not present in any other equation. Formally, the set of assignment functions which meet the constraint of equation A, B_i but not all others is not empty:

$$\left\{ \vec{B} : (\vec{B}(A) = B_i) \bigwedge_{\forall j=1,...,n-1; j\neq i} (\vec{B}(A) \neq B_j) \bigwedge_{\forall A' \neq A; \forall j=1,...,n-1} (\vec{B}(A') \neq B_j) \right\} \neq \emptyset. \tag{30}$$

But the assignment functions $\vec{B}$ such that $\vec{B}(A) = B_i$ and $\forall A' \neq A$ $\vec{B}(A') = \Theta$ all meet condition (30). Indeed they obviously meet $\vec{B}(A) \neq B_j$ for all $j \neq i$ while clearly for all $A' \subseteq \Theta$ $\vec{B}(A') = \Theta \neq B_j$, as $j < n$ so that $B_j \neq \Theta$.

A non-negative solution of (29) (and hence of (28)) can be obtained by setting for each equation one of such variables equal to the first member $\alpha^A_{B_i}$, and all the others to zero.

Proof of Theorem 7

The proof is divided in two parts.

1. We first need to find an assignment $\vec{B} : 2^\Theta \to \mathcal{C}_\rho$ which generates co^ρ. Each singleton x_i is mapped by ρ to the position j: $i = \rho(j)$. Then, given any event $A = \{x_{i_1}, ..., x_{i_m}\}$ its elements are mapped to the new positions $x_{j_{i_1}}, ..., x_{j_{i_m}}$, where $i_1 = \rho(j_{i_1}), ..., i_m = \rho(j_{i_m})$. But then the map

$$\vec{B}_\rho(A) = \vec{B}_\rho(\{x_{i_1}, ..., x_{i_m}\}) = S^\rho_j \doteq \{x_{\rho(1)}, ..., x_{\rho(j)}\}$$

where

$$j \doteq \max\{j_{i_1}, ..., j_{i_m}\}$$

maps each event A to the smallest S^ρ_i in the chain which contains A: $j = \min\{i : A \subseteq S^\rho_i\}$. Therefore it generates a co.b.f. with b.p.a. (11), i.e. co^ρ.

2. In order for co^ρ to be an actual vertex, we need to ensure that it cannot be written as a convex combination of the other (pseudo) vertices $o^{\vec{B}}[b]$:

$$co^\rho = \sum_{\vec{B} \neq \vec{B}_\rho} \alpha_{\vec{B}} o^{\vec{B}}[b], \qquad \sum_{\vec{B} \neq \vec{B}_\rho} \alpha_{\vec{B}} = 1, \quad \forall \vec{B} \neq \vec{B}_\rho \ \alpha_{\vec{B}} \geq 0.$$

As $m_{o^{\vec{B}}}(B_i) = \sum_{A:\vec{B}(A)=B_i} m_b(A)$ the above condition reads as

$$\left\{ \sum_{A \subseteq B_i} m_b(A) \left(\sum_{\vec{B}:\vec{B}(A)=B_i} \alpha_{\vec{B}} \right) = \sum_{A \subseteq B_i : \vec{B}_\rho(A) = B_i} m_b(A) \qquad \forall B_i \in \mathcal{C}. \right.$$

Remembering that $\vec{B}_\rho(A) = B_i$ iff $A \subseteq B_i, \not\subset B_{i-1}$ we get

$$\left\{ \sum_{A \subseteq B_i} m_b(A) \left(\sum_{\vec{B}:\vec{B}(A)=B_i} \alpha_{\vec{B}} \right) = \sum_{A \subseteq B_i, \not\subset B_{i-1}} m_b(A) \qquad \forall B_i \in \mathcal{C}. \right.$$

For $i = 1$ the condition is $m_b(B_1) \left(\sum_{\vec{B}:\vec{B}(B_1)=B_1} \alpha_{\vec{B}} \right) = m_b(B_1)$ i.e.

$$\sum_{\vec{B}:\vec{B}(B_1)=B_1} \alpha_{\vec{B}} = 1, \qquad \sum_{\vec{B}:\vec{B}(B_1)\neq B_1} \alpha_{\vec{B}} = 0.$$

Replacing this condition in the second constraint $i = 2$ yields

$$m_b(B_2 \setminus B_1) \left(\sum_{\substack{\vec{B} : \vec{B}(B_1) = B_1, \\ \vec{B}(B_2 \setminus B_1) = B_2}} \alpha_{\vec{B}} \right) + m_b(B_2) \left(\sum_{\substack{\vec{B} : \vec{B}(B_1) = B_1, \\ \vec{B}(B_2) = B_2}} \alpha_{\vec{B}} \right) = \\ = m_b(B_2 \setminus B_1) + m_b(B_2)$$

i.e.

$$m_b(B_2 \setminus B_1) \left(\sum_{\substack{\vec{B} : \vec{B}(B_1) = B_1, \\ \vec{B}(B_2 \setminus B_1) \neq B_2}} \alpha_{\vec{B}} \right) + m_b(B_2) \left(\sum_{\substack{\vec{B} : \vec{B}(B_1) = B_1, \\ \vec{B}(B_2) \neq B_2}} \alpha_{\vec{B}} \right) = 0$$

which implies $\alpha_{\vec{B}} = 0$ for all the assignment functions $\vec{B}$ such that $\vec{B}(B_2 \setminus B_1) \neq B_2$ or $\vec{B}(B_2) \neq B_2$. The only non-zero coefficients can then be the $\alpha_{\vec{B}}$ s.t. $\vec{B}(B_1) = B_1$, $\vec{B}(B_2 \setminus B_1) = B_2$, $\vec{B}(B_2) = B_2$.
By induction you get that $\forall \vec{B} \neq \vec{B}_\rho$ we have $\alpha_{\vec{B}} = 0$.

References

[1] R. R. Murphy, Dempster-Shafer theory for sensor fusion in autonomous mobile robots, IEEE Transactions on Robotics and Automation 14 (1998) 197–206.

[2] S. McClean, B. Scotney, Using evidence theory for the integration of distributed databases, International Journal of Intelligent Systems 12 (1997) 763–776.

[3] A. Ayoun, P. Smets., Data association in multi-target detection using the transferable belief model, Intern. J. Intell. Systems.

[4] B. Ristic, P. Smets, The TBM global distance measure for the association of uncertain combat ID declarations, Information Fusion 7 (3) (2006) 276–284.

[5] G. Shafer, A Mathematical Theory of Evidence, Princeton University Press, 1976.

[6] A. P. Dempster, Upper and lower probability inferences based on a sample from a finite univariate population, Biometrika 54 (1967) 515–528.

[7] F. Cuzzolin, Geometry of upper probabilities, in: Proceedings of ISIPTA'03, Lugano, Switzerland, 2003.

[8] F. Cuzzolin, A geometric approach to the theory of evidence, IEEE Transactions on Systems, Man and Cybernetics part C 38 (4) (2008) 522–534.

[9] D. Dubois, H. Prade, Possibility theory, Plenum Press, New York, 1988.

[10] D. Dubois, H. Prade, Consonant approximations of belief functions, International Journal of Approximate Reasoning 4 (1990) 419–449.

[11] C. Joslyn, G. Klir, Minimal information loss possibilistic approximations of random sets, in: Proc. 1992 FUZZ-IEEE Conference, San Diego, 1992, pp. 1081–1088.

[12] D. Dubois, H. Prade, S. Sandri, On possibility/probability transformations, in: R. Lowen, M. Roubens (Eds.), Fuzzy Logic: State of the Art, Kluwer Academic Publ., 1993, pp. 103–112.

[13] P. Baroni, Extending consonant approximations to capacities, in: Proceedings of IPMU, 2004, pp. 1127–1134.

[14] S. Maass, A philosophical foundation of non-additive measure and probability, Theory and decision 60 (2006) 175–191.

[15] V. Danilov, G. Koshevoy, Cores of cooperative games, superdifferentials of functions and the minkovski difference of sets, Journal of Mathematical Analysis Applications 247 (2000) 1–14.

[16] G. Koshevoy, Distributive lattices and products of capacities, Journal of Mathematical Analysis Applications 219 (1998) 427–441.

[17] P. Black, An examination of belief functions and other monotone capacities, PhD dissertation, Department of Statistics, Carnegie Mellon University (1996).

[18] P. Black, Geometric structure of lower probabilities, in: Goutsias, Malher, Nguyen (Eds.), Random Sets: Theory and Applications, 1997, pp. 361–383.

[19] V. Ha, P. Haddawy, Geometric foundations for interval-based probabilities, in: A. Cohn, L. Schubert, S. Shapiro (Eds.), KR'98: Principles of Knowledge Representation and Reasoning, Morgan Kaufmann, San Francisco, California, 1998, pp. 582–593.

[20] R. R. Yager, Class of fuzzy measures generated from a Dempster-Shafer belief structure, International Journal of Intelligent Systems 14 (1999) 1239–1247.

[21] C. Roemer, A. Kandel, Applicability analysis of fuzzy inference by means of generalized Dempster-Shafer theory, IEEE Transactions on Fuzzy Systems 3 (4) (1995) 448–453.

[22] G. J. Klir, W. Zhenyuan, D. Harmanec, Constructing fuzzy measures in expert systems, Fuzzy Sets and Systems 92 (1997) 251–264.

[23] S. Heilpern, Representation and application of fuzzy numbers, Fuzzy Sets and Systems 91 (1997) 259–268.

[24] L. Caro, A. B. Nadjar, Generalization of the Dempster-Shafer theory: a fuzzy-valued measure, IEEE Transactions on Fuzzy Systems 7 (1999) 255–270.

[25] P. Smets, The transferable belief model and possibility theory, in: K. Y. (Ed.), Proceedings of NAFIPS-90, 1990, pp. 215–218.

[26] C. Joslyn, Possibilistic normalization of inconsistent random intervals, Advances in Systems Science and Applications (1997) 44–51.

[27] B. Dubrovin, S. Novikov, A. Fomenko, Sovremennaja geometrija. Metody i prilozenija, Nauka, Moscow, 1986.

[28] C. Joslyn, Towards an empirical semantics of possibility through maximum uncertainty, in: R. Lowen, M. Roubens (Eds.), Proc. IFSA 1991, Vol. A, 1991, pp. 86–89.

[29] P. Smets, R. Kennes, The transferable belief model, Artificial Intelligence 66 (1994) 191–234.

[30] F. Cuzzolin, Moebius inverses of plausibility and commonality functions and their geometric interpretation, submitted to the Internation Journal on Uncertainty, Fuzziness, and Knowledge-Based Systems.

[31] R. Yager, The entailment principle for Dempster-Shafer granules, International Journal of Intelligent Systems 1 (1986) 247–262.

[32] D. Dubois, H. Prade, A set-theoretic view of belief functions: Logical operations and approximations by fuzzy sets, International Journal of General Systems 12 (1986) 193–226.

[33] M. Bruning, D. Denneberg, Max-min σ-additive representation of monotone measures, Statistical Papers 34 (2002) 23–35.