GEOMETRY OF DEMPSTER'S RULE

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ABSTRACT

In this paper we analyze the nature of Shafer's belief functions as geometric entities. We focus on the geometric behavior of Dempster's rule of combination in the context of the *belief space*, i.e. the set $\mathcal{S}$ of all the admissible belief functions defined over a given finite domain. After showing how convex closure and orthogonal sum commute in the belief space and introducing the concept of *conditional subspace*, we describe the *local* behavior of the rule of combination as internal operation of $\mathcal{S}$ by means of the language borrowed from convex geometry.

1. INTRODUCTION

The *theory of evidence* ([8]) is perhaps one of the most successful theories of probable reasoning, proposed in the late Seventies as an alternative to the classical Bayesian formalism. In a series of recent works we introduced a geometrical interpretation of the notion of generalized probability or *belief function* ([8]). Once a coordinate in a Cartesian reference frame in the Euclidean space $\mathbb{E}^N$ is assigned to each subset of a given finite domain Θ, there exists a 1-1 correspondence between belief functions with domain Θ and points of $\mathbb{E}^N$. The collection $\mathcal{S}_\Theta$ of all the points of $\mathbb{E}^N$ corresponding to a belief function is called *belief space*.
In [3] we have proved that this belief space has the form of a *simplex*, where the collection of probabilities $\mathcal{P}$ definable over the same frame is a proper subset of its boundary: $\mathcal{P} \subseteq \partial\mathcal{S}$.

In this framework it is natural to consider the orthogonal sum [4] of a pair of belief functions as an internal operator in $\mathcal{S}$. In this paper we will actually describe the geometric behavior of Dempster's rule in the belief space. We will first analyze its connections to the convex closure operator as mathematical transposition of the basic probability assignment, and depict its global action in terms of *conditional subspaces*. Next, we will prove that it is possible to reduce Dempster's sums to convex sums of Bayes' conditional functions. Finally, a better comprehension of the local geometry of the orthogonal sum will be achieved through the analysis of the shape of constant mass loci.

1.1. Previous work

To our knowledge the first attempts of studying the geometric properties of the belief functions are due to the author ([3]), even if some work has been done on the convex structure of classical probabilities.
The closest reference is perhaps a recent paper of Ha and Haddawy [5] where they exploit methods of convex geometry to represent probability intervals in a computationally efficient fashion, by means of a data structure called *pcc-tree*, founded on a generalization of the convex closure called cc-operator. They show that belief functions can be represented by 2-level pcc-trees and study the evidential updating in this context. More generally, an impressively large bibliography ([6], [7]) is dedicated to the study of automated reasoning, and in particular to Dempster's rule as a way of pooling new evidence.

2. EVIDENTIAL REASONING

Following Shafer [8] let us call the finite set of possible outcomes for a decision problem *frame of discernment*.

Definition 1. *A* basic probability assignment *(b.p.a.) over a frame* Θ *is a function* $m : 2^\Theta \to [0,1]$ *such that*

$$m(\emptyset) = 0, \ \sum_{A\subset\Theta} m(A) = 1.$$

Subsets of Θ associated to non-zero values of m are called *focal elements* and their union $\mathcal{C}$ *core*.
Now suppose a b.p.a. is introduced on an arbitrary frame.

Definition 2. *The* belief function *(b.f.)* s *induced by the basic probability assignment* m *is defined as:*

$$s(A) = \sum_{B\subset A} m(B).$$

Conversely, the basic probability assignment m associated to a given belief function s can be uniquely recovered by means of the *Moebius inversion formula*

$$m(A) = \sum_{B\subset A} (-1)^{|A-B|} s(B) \qquad (1)$$

so that there is a 1-1 correspondence between the set functions $m \leftrightarrow s$. We will often denote by m_s the b.p.a. corresponding to s. When all the focal elements are points of Θ ($m(B) = 0$ if $|B| > 1$) we get a classical probability.

Belief functions representing distinct bodies of evidence can be combined together by means of the so called *Dempster's rule of combination* ([4]).

Definition 3. *The* Dempster combination *or* orthogonal sum $s_1 \oplus s_2$ *of two belief functions is a new belief function whose focal elements are all the possible intersections between the combining focal elements and whose b.p.a. is given by*

$$m(A) = \frac{\sum_{i,j:A_i \cap B_j = A} m_1(A_i) m_2(B_j)}{1 - \sum_{i,j:A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j)}. \qquad (2)$$

When all the intersections between focal elements of the two functions are empty, the denominator of equation (2) (*normalization factor*) goes to zero and we say that s_1 and s_2 are *not combinable*.

3. BELIEF SPACE

Now consider a frame of discernment Θ and introduce in the Euclidean space $\mathbb{R}^{|2^\Theta|-1}$ an orthonormal reference frame $\{x_i\}_{i=1,...,|2^\Theta|-1}$ in which, given an arbitrary ordering in $2^\Theta \setminus \{\emptyset\}$, each coordinate function x_i measures the belief value associated to a the i-th subset of Θ.

Definition 4. *The* belief space *associated to Θ is the set of points $\mathcal{S}_\Theta$ of $\mathbb{R}^{|2^\Theta|-1}$ corresponding to belief functions.*

From now on, we will assume the domain Θ fixed, and denote the belief space by $\mathcal{S}$. To determine which points $s \in \mathbb{R}^{|2^\Theta|-1}$ correspond to a belief function we can exploit the Moebius inversion lemma (1), by simply computing the corresponding b.p.a. and checking the axioms m must obey. The *normalization* constraint $\sum_{A \subset \Theta} m(A) = 1$ trivially translates into $\mathcal{S} \subset \{s : s(\Theta) = 1\}$. The *positivity* condition is far more interesting, generating an inequality that resounds the third axiom of belief functions ([8], page 5): $\forall A \subset \Theta$

$$s(A) - \sum_{B \subset A, |B|=|A|-1} s(B) + ... + (-1)^{|A-B|} \cdot$$
$$\cdot \sum_{|B|=k} s(B) + ... + (-1)^{|A|-1} \sum_{\theta \in \Theta} s(\{\theta\}) \geq 0. \qquad (3)$$

It is not difficult to prove by means of equation (3) (see [1] for the details) that $\mathcal{S}$ is *convex*. Furthermore, we can give an exact expression for the belief space.

Let us denote with

$$P_A \doteq s \in \mathcal{S} \; s.t. \; m_s(A) = 1, \; m_s(B) = 0 \; B \neq A$$

the unique belief function assigning all the mass to a single subset A of Θ. It can be proved that (see [3] again), denoting by $\mathcal{E}_s$ the list of focal elements of s,

Theorem 1. *The set of all the belief functions with focal elements in a given collection X is closed and convex in $\mathcal{S}$:*

$$\{s : A \in \mathcal{E}_s \Rightarrow A \in X\} = Cl(\{P_A : A \in X\}).$$

The following is just a trivial consequence.

Corollary 1. *The belief space $\mathcal{S}$ coincides with the convex closure of all the simple b.f.s P_A,*

$$\mathcal{S} = Cl(P_A, \; A \subset \Theta, \; A \neq \emptyset). \qquad (4)$$

More precisely, any belief function $s \in \mathcal{S}$ can be written as a convex sum:

$$s = \sum_{A \subset \Theta, \, A \neq \emptyset} m_s(A) \cdot P_A. \qquad (5)$$

4. COMMUTATIVITY

The convex closure operator is then strictly related to the notion of belief function, since it reflects the way a basic probability assignment generates it. We now have two different operations acting on belief functions in the belief space, convex closure and Dempster's rule, but still do not know if they have any meaningful interaction.

Theorem 2. *If $\sum_i \alpha_i = 1$, then*

$$s \oplus \sum_i \alpha_i s_i = \sum_i \beta_i (s \oplus s_i), \quad \beta_i = \frac{\alpha_i \Delta_i}{\sum_{i=1}^n \alpha_i \Delta_i}$$

where Δ_i is the normalization factor for the i-th combination $s \oplus s_i$.

Proof. (sketch) For $s \oplus \sum_i \alpha_i s_i$ Dempster's rule yields

$$m_{s \oplus \sum_i \alpha_i s_i}(A) = \frac{\sum_{A_k \cap B_j = A} m_{\sum_i \alpha_i s_i}(A_k) m_s(B_j)}{1 - \sum_{A_k \cap B_j = \emptyset} m_{\sum_i \alpha_i s_i}(A_k) m_s(B_j)} =$$

$$= \frac{\sum_{A_k \cap B_j = A} (\sum_i \alpha_i m_{s_i}(A_k)) \cdot m_s(B_j)}{1 - \sum_{A_k \cap B_j = \emptyset} (\sum_i \alpha_i m_{s_i}(A_k)) \cdot m_s(B_j)} =$$

$$= \frac{\sum_i \alpha_i \cdot \sum_{A_k \cap B_j = A} m_{s_i}(A_k) m_s(B_j)}{\sum_i \alpha_i \cdot (1 - \sum_{A_k \cap B_j = \emptyset} m_{s_i}(A_k) m_s(B_j))}.$$

Since for $A_k \notin \mathcal{E}_{s_i}$ $m_{s_i}(A_k)$ vanishes, we are left for each i with the focal elements of s_i:

$$\frac{\sum_i \alpha_i \cdot \sum_{E_k \cap B_j = A} m_{s_i}(E_k) m_s(B_j)}{\sum_i \alpha_i \cdot (1 - \sum_{E_k \cap B_j = \emptyset} m_{s_i}(E_k) m_s(B_j))}, \quad E_k \in \mathcal{E}_{s_i}.$$

Finally, we just need to remember that

$$m_{s\oplus s_i}(A) = \frac{\sum_{E_k\cap B_j=A} m_{s_i}(E_k)m_s(B_j)}{1-\sum_{E_k\cap B_j=\emptyset} m_{s_i}(E_k)m_s(B_j)} \doteq \frac{N_i(A)}{\Delta_i}$$

so that, called $N_i(A)$ the numerator of the above expression,

$$m_{s\oplus\sum_i \alpha_i s_i}(A) = \frac{\sum_i \alpha_i N_i(A)}{\sum_i \alpha_i \Delta_i} = \frac{\sum_i \alpha_i \Delta_i \cdot m_{s\oplus s_i}(A)}{\sum_i \alpha_i \Delta_i}$$

and we have the thesis. □

For a complete proof see [2]. Note that, since $\sum_i \beta_i = 1$, any combination with a convex sum of belief functions is also a convex sum of all the partial combinations. Let us now introduce the notation $v(x_1,...,x_n)$, indicating the smallest vector space in $\mathbb{R}^{(\cdot)}$ containing a number of given points $x_1,...,x_n$. Namely

$$v(x_1,...,x_n) \doteq \{x : x = \sum_{i=1}^n \alpha_i x_i, \quad \sum_i \alpha_i = 1\}.$$

Theorem 3. *$\oplus$ and $v(.)$ commute, i.e. if s is combinable with s_i $\forall i = 1,...,n$, then*

$$s \oplus v(s_1,...,s_n) = v(s\oplus s_1,...,s\oplus s_n).$$

Proof. We need to prove that, given the Δ_i's and any set of coefficients $\{\beta_i,\ i=1,...,n\}$ such that $\sum_i \beta_i = 1$ there exists another collection of coefficients $\{\alpha_i,\ i=1,...,n\}$ with $\sum_i \alpha_i = 1$ such that $\beta_i = \frac{\alpha_i\Delta_i}{\sum_i \alpha_i \Delta_i}$. Now, if we take $\tilde{\alpha}_i \doteq \beta_i/\Delta_i$ we get $\beta_i = \beta_i/\sum_i \beta_i = \beta_i$, and the system is satisfied up to the normalization constraint. We then just normalize, by choosing

$$\alpha_i = \tilde{\alpha}_i / \sum_i \tilde{\alpha}_i = \frac{\beta_i}{\Delta_i \sum_i (\frac{\beta_i}{\Delta_i})}.$$

□

As a straightforward consequence, since

$$Cl(x_1,...,x_n) = \{x : x = \sum_{i=1}^n \alpha_i x_i,\ \sum_i \alpha_i = 1,\ \alpha_i \geq 0\},$$

Corollary 2. *Cl and $\oplus$ commute, i.e. if s is combinable with s_i $\forall i = 1,...,n$, then*

$$s \oplus Cl(s_1,...,s_n) = Cl(s\oplus s_1,...,s\oplus s_n).$$

5. CONDITIONAL SUBSPACES

The commutativity results we proved above are powerful tools. They allow us to define geometric counterparts of the notions of combinability and conditioning.

Definition 5. *The* conditional subspace $\langle s\rangle$ *is the set of belief functions conditioned by a given function s, namely*

$$\langle s\rangle \doteq \{s\oplus t,\ t\in\mathcal{S}\ s.t.\ \exists\ s\oplus t\}. \qquad (6)$$

In a sense, its conditional subspace is the possible "future" of s when combined with new evidence by means of Dempster's rule. Since not every belief function is combinable with an arbitrary s, we first need to understand the geometric structure of combinable functions.

Definition 6. *The* compatible subspace $C(s)$ *associated to a belief function s is the collection of all the b.f.s with focal elements included in the core of s:*

$$C(s) \doteq \{s' : \mathcal{C}_{s'} \subset \mathcal{C}_s\}.$$

Theorem 4. $\langle s\rangle = s\oplus C(s) = Cl\{s\oplus P_A,\ A\subset \mathcal{C}_s\}$.

Note that $s\oplus P_{\mathcal{C}_s} = s$, hence s is always a vertex of $\langle s\rangle$. Of course $\langle s\rangle \subset C(s)$, since the core is a *monotone function* on the poset $(\mathcal{S}, \geq_\oplus)$.

6. CONVEX FORM OF DEMPSTER'S RULE

We know from corollary 1 that any belief function $s\in\mathcal{S}$ can be written as a convex sum of simple b.f.s. By applying theorem 2 to equation (5) we get

$$s\oplus\sigma = \sum_A \mu(A)\cdot s\oplus P_A, \quad \mu(A) = \frac{m_\sigma(A)\Delta_A}{\sum_i m_\sigma(A)\Delta_A}$$

where Δ_A is the normalization factor for $s\oplus P_A$, equal to

$$\Delta_A = \sum_{B:B\cap A\neq\emptyset} m_s(B) = \sum_{B\not\subset A^c} m_s(B) =$$
$$= 1 - \sum_{B\subset A^c} m_s(B) = 1 - s(A^c) = P^*_s(B)$$

a quantity called *upper probability* of B. In other words

Theorem 5.

$$s\oplus\sigma = \sum_{A\subset\Theta,\ A\neq\emptyset} \frac{m_\sigma(A)P^*_s(A)}{\sum_B m_\sigma(B)P^*_s(B)}\cdot s\oplus P_A \qquad (7)$$

the application of Dempster's rule to a belief function yields a new b.f. whose coefficients are weighted by the upper probabilities of the events in terms of the new evidence s. We can also notice that

Proposition 1. $P^*_s(A) = 0 \Leftrightarrow \nexists s\oplus P_A$.

Proof. In fact, $P^*_s(A) = 0$ when

$$1 - s(A^c) = 0 \Rightarrow s(A^c) = 1 \Rightarrow A^c \supset \mathcal{C}_s \Rightarrow A\cap\mathcal{C}_s = \emptyset$$

while $s\oplus P_A$ does not exists when $A\cap A_i = \emptyset\ \ \forall A_i\in\mathcal{E}_s$ i.e. $A\cap\mathcal{C}_s = \emptyset$. □

so that expression (7) is valid no matter what the focal elements of s are. We can further simplify the convex formulation of Dempster's rule by noticing that some of the partial combinations $s \oplus P_A$ may coincide. In fact,

$$s \oplus P_A = s \oplus P_B \quad \Leftrightarrow \quad A \cap \mathcal{C}_s = B \cap \mathcal{C}_s.$$

Hence we can write

$$\begin{array}{l} s \oplus \sigma = \sum_{A \subset \mathcal{C}_s} s \oplus P_A \cdot (\sum_{B \cap \mathcal{C}_s = A} \mu(B)) = \\ = \displaystyle\sum_{A \subset \mathcal{C}_s} s \oplus P_A \cdot \frac{\sum_{B \cap \mathcal{C}_s = A} m_\sigma(B) P^*_s(B)}{\sum_{B \subset \Theta} m_\sigma(B) P^*_s(B)}. \end{array} \tag{8}$$

Equation (7) has an interesting probabilistic interpretation. Combinations of the form $s \oplus P_A$ can be thought of as applications of Bayes' rule to the given belief function s, when conditioning with respect to the event A, $s|A \doteq s \oplus P_A$. This way, proposition 5 can be interpreted as a decomposition of Dempster's rule of combination in terms of the classical Bayes' rule of conditioning.

7. CONSTANT MASS LOCI

To better understand the local geometric behavior of Dempster's rule it can be useful to analyze the case of the simplest, binary frame $\{x, y\}$. Figure 1 represents the belief space $\mathcal{S}_2$ and the conditional subspace associated to an arbitrary b.f. s_1, with $s(\{x\})$ on the x axis and $s(\{y\})$ on the y axis ($s(\Theta) = 1 = const$ is neglected). Given another b.f.

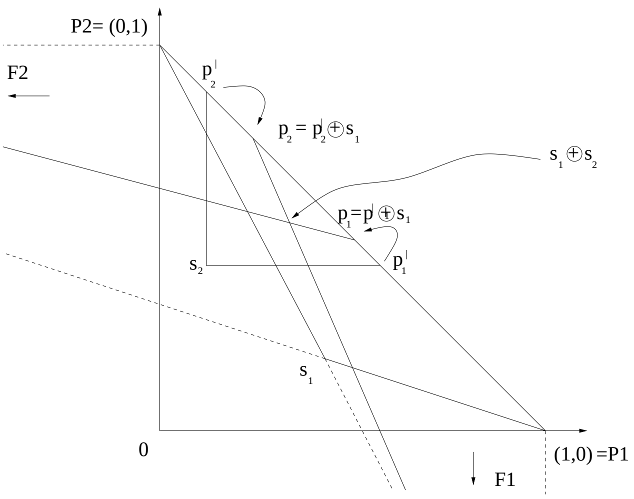

Figure 1: Graphical construction of Dempster's orthogonal sum in $\mathcal{S}_2$.

$s_2 = (a, b)$, their combination $s_1 \oplus s_2$ can be computed by intersecting the images of the loci $m(x) = a$, $m(y) = b$ through the map $s_1 \oplus (.)$.

All these images prove to lay on lines with a single point as common intersection (respectively F_1 and F_2 in the figure).

According to the intuition provided by the example, let us define

$$\mathcal{H}^k_A \doteq \{s : m_s(A) = k\}.$$

The dimension of $\mathcal{H}^k_A$ is of course $dim(\mathcal{S}) - 1$. For $\mathcal{S}_2$ we get $dim(\mathcal{H}^k_A) = 4 - 2 - 1 = 1$ so that any constant mass locus is a line.

Theorem 6. *If $s_1, ..., s_n$ assign an equal mass k to a subset A, then any convex combination of them $s \in v(s_1, ..., s_n)$ assigns the same basic probability k to A.*

Proof. Let us suppose a number of b.f.s $s_1, ..., s_n$ satisfy the additional constraint $m_{s_i}(A) = k$. By using the Moebius inversion lemma (equation 1) we get, given a set of coefficients $\alpha_i,\ i = 1, ..., n$ such that $\sum_i \alpha_i = 1$,

$$\begin{array}{l} m_{\sum_i \alpha_i s_i} = \displaystyle\sum_{B \subset A} (-1)^{|B-A|} \sum_i \alpha_i s_i(A) = \\ = \displaystyle\sum_i \sum_{B \subset A} (-1)^{|B-A|} \alpha_i s_i(A) = \\ = \displaystyle\sum_i \alpha_i \sum_{B \subset A} (-1)^{|B-A|} s_i(A) = \sum_i \alpha_i m_i(A) \end{array}$$

but, since $m_i(A) = k\ \forall i$, $\sum_i \alpha_i m_i(A) = \sum_i \alpha_i k = k \cdot \sum_i \alpha_i = k$. □

Almost immediately we have that [2]

Theorem 7. *The constant mass locus $\mathcal{H}^k_A$ can be expressed as a convex closure*

$$\mathcal{H}^k_A = k \cdot P_A + (1 - k) \cdot Cl(P_B : B \subset \Theta,\ B \neq A). \tag{9}$$

7.1. Action of Dempster's rule on $\mathcal{H}^k_A$

We can now easily exploit the commutativity property to compute images of constant mass loci through the orthogonal sum, obtaining (if $C(\varsigma) = \mathcal{S}$)

$$\varsigma \oplus \mathcal{H}^k_A = Cl(\varsigma \oplus [k \cdot P_A + (1 - k) \cdot P_B] : B \subset \Theta, B \neq A)$$

where $\varsigma \oplus k \cdot P_A$ is given by

$$\frac{kP^*_s(A)}{1 - k + kP^*_s(A)} \cdot s \oplus P_A + \frac{1 - k}{1 - k + kP^*_s(A)} \cdot s \tag{10}$$

while in general

$$\varsigma \oplus [kP_A + (1 - kP_B)] = \beta_A \cdot \varsigma \oplus P_A + \beta_B \cdot \varsigma \oplus P_B \tag{11}$$

with

$$\beta_A = \frac{kP^*_s(A)}{kP^*_s(A) + (1-k)P^*_s(B)},\ \beta_B = \frac{(1-k)P^*_s(B)}{kP^*_s(A) + (1-k)P^*_s(B)}.$$

Of course when $k = 0$ we have

$$\varsigma \oplus \mathcal{H}^0_A = Cl(\varsigma, \varsigma \oplus P_B : B \neq A)$$

i.e. we get the *antipodal face* [1] of $\langle \varsigma \rangle$ with respect to A.

8. GEOMETRIC ORTHOGONAL SUM

The intuition provided by the two-dimensional belief space suggests that

Proposition 2. *The family of subspaces* $\{v(s \oplus \mathcal{H}_A^k) : 0 \leq k < 1\}$ *has a non-empty common intersection*[1].

as a formal proof [2] confirms. We can hence introduce the notion of *foci* of a conditional subspace.

Definition 7. *We call A-th* focus *of the conditional subspace $\langle s \rangle$ the linear variety*

$$\mathcal{F}_A \doteq \bigcap_{k\in[0,1)} v(s\oplus\mathcal{H}_A^k) = \bigcap_{k\in[0,1)} v(\{s\oplus\sigma : m_\sigma(A) = k\}).$$

It can be proved that (see [2] again for a complete proof)

Theorem 8. *The A-th focus of the conditional subspace $\langle s \rangle$ can be expressed as a vector subspace*

$$\mathcal{F}_A = v(\varsigma_B, B \neq A, \Theta) \qquad (12)$$

generated by a collection of $2^{|\Theta|} - 3$ focal points

$$\varsigma_B = \frac{1}{1 - P_s^*(B)} \cdot s + \frac{P_s^*(B)}{P_s^*(B) - 1} \cdot s \oplus P_B. \qquad (13)$$

Although these focal points do not belong to the belief space, they possess a very intuitive meaning in terms of mass assignment.

Theorem 9.

$$\varsigma_B = \lim_{k\to+\infty} s\oplus(kP_\Theta+(1-k)P_B) = \lim_{k\to+\infty} s\oplus(1-k)P_B.$$

Proof.

$$\lim_{k\to+\infty} s \oplus (1-k)P_B = \lim_{k\to+\infty} [kP_\Theta + (1-k)P_B] =$$

$$\lim_{k\to+\infty}\left(\frac{k}{k+(1-k)P_s^*(B)} s + \frac{(1-k)P_s^*(B)}{k+(1-k)P_s^*(B)} s \oplus P_B\right) = \varsigma_B.$$

□

At this point a possible geometric algorithm for the orthogonal sum $s \oplus \varsigma$ is easily outlined.

Algorithm.

1. First all the foci $\{\mathcal{F}_A, \forall A \subset \Theta\}$ of the subspace $\langle s \rangle$ conditioned by the first belief function are computed, by calculating the corresponding focal points (13);
2. then, an additional point (10) for each subset A is detected, selecting the subspace $v(s \oplus \mathcal{H}_A^{m_\varsigma(A)})$;
3. all these subspaces are then intersected, eventually yielding the desired combination.

Note that the focal points can be computed *just once*, since they are trivial functions of the upper probabilities $P_s^*(B)$ $\forall B \subset \Theta$ associated to the belief function chosen as basis. After that, different groups of points are selected for each foci with no need for further calculations.

[1] k=1 cannot be considered for $v(s \oplus \mathcal{H}_A^1) \cap \mathcal{S} = s \oplus P_A$. The problem can be avoided by considering the extension of Dempster's rule to sum functions

9. CONCLUSIONS

In this paper we described the geometric properties of Dempster's rule of combination in the context of the belief space. We illustrated its major connections with the convex closure operator and unveiled an interesting convex decomposition of the orthogonal sum in terms of Bayes' conditioning. We also investigated the geometry of conditional subspaces, eventually proposing a geometric construction for the orthogonal sum.

10. REFERENCES

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