# Complexes of outer consonant approximations

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**Abstract.** In this paper we discuss the problem of approximating a belief function (b.f.) with a necessity measure or "consonant belief function" (co.b.f.) from a geometric point of view. We focus in particular on outer consonant approximations, i.e. co.b.f.s less committed than the original b.f. in terms of degrees of belief. We show that for each maximal chain of focal elements the set of outer consonant approximation is a polytope. We describe the vertices of such polytope, and characterize the geometry of maximal outer approximations.

## 1 Introduction

The theory of evidence (ToE) [10] is a popular approach to uncertainty description. Probabilities are there replaced by *belief functions* (b.f.s), which assign values between 0 and 1 to subsets of the sample space $\Theta$ instead of single elements. Possibility theory [4], on its side, is based on *possibility measures*, i.e., functions $Pos : 2^\Theta \rightarrow [0,1]$ on $\Theta$ such that $Pos(\bigcup_i A_i) = \sup_i Pos(A_i)$ for any family $\{A_i | A_i \in 2^\Theta, i \in I\}$ where $I$ is an arbitrary set index. Given a possibility measure $Pos$, the dual *necessity* measure is defined as $Nec(A) = 1 - Pos(A)$.
Necessity measures have as counterparts in the theory of evidence *consonant* b.f.s, i.e. belief functions whose focal elements are nested [10]. The problem of approximating a belief function with a necessity measure is then equivalent to approximating a belief function with a consonant b.f. [5, 9, 8, 1]. As possibilities are completely determined by their values on the singletons $Pos(x)$, $x \in \Theta$, they are less computationally expensive than b.f.s, making the approximation process interesting for many applications. The points of contact between evidence (in the transferable belief model implementation) and possibility theory have been for instance investigated by Ph. Smets [11].

A geometric interpretation of uncertainty theory has been recently proposed [2] in which several classes of uncertainty measures (among which belief functions and possibilities) are represented as points of a Cartesian space.
In this paper we consider the problem of approximating a belief function with a possibility/necessity [5] from such geometric point of view. We focus in particular on the class of *outer consonant approximations* of belief functions.
More precisely, after reviewing the basic notions of evidence and possibility theory we formally introduce the consonant approximation problem, and in particular the notion of outer consonant approximation. We then recall how the set of all consonant belief functions forms a *simplicial complex*, a structured collection

of higher-dimensional triangles or "simplices". Each such maximal simplex is associated with a maximal chain of subsets of $\Theta$. Starting from the simple binary case we prove that the set of outer consonant approximations of a b.f. forms, on each such maximal simplex, a polytope. We investigate the form of its vertices and prove that one of them corresponds to the maximal outer approximation, the one [5] generated by a permutation of the element of $\Theta$. To improve the readability of the paper all major proofs are collected in an Appendix. Illustrative examples accompany all the presented results.

## 2 Outer consonant approximations of belief functions

**Belief and possibility measures.** A *basic probability assignment* (b.p.a.) over a finite set *(frame of discernment [10])* $\Theta$ is a function $m : 2^\Theta \to [0,1]$ on its power set $2^\Theta = \{A \subseteq \Theta\}$ such that $m(\emptyset) = 0$, $\sum_{A\subseteq\Theta} m(A) = 1$, and $m(A) \geq 0$ $\forall A \subseteq \Theta$. Subsets of $\Theta$ associated with non-zero values of $m$, $\{E \subset \Theta : m(E) \neq 0\}$ are called *focal elements*. The *belief function* $b : 2^\Theta \to [0,1]$ associated with a basic probability assignment $m$ on $\Theta$ is defined as: $b(A) = \sum_{B\subseteq A} m(B)$. The *plausibility function* (pl.f.) $pl_b : 2^\Theta \to [0,1]$, $A \mapsto pl_b(A)$ such that $pl_b(A) \doteq 1 - b(A^c) = \sum_{B\cap A\neq\emptyset} m_b(B)$ expresses the amount of evidence *not against* $A$. A probability function is simply a peculiar belief function assigning non-zero masses to singletons only (*Bayesian* b.f.): $m_b(A) = 0$ $|A| > 1$. A b.f. is said to be *consonant* if its focal elements $\{E_i, i = 1, ..., m\}$ are nested: $E_1 \subset E_2 \subset ... \subset E_m$. It can be proven that [4, 7] the plausibility function $pl_b$ associated with a belief function $b$ on a domain $\Theta$ is a possibility measure iff $b$ is consonant. Equivalently, a b.f. $b$ is a necessity iff $b$ is consonant.

**Outer consonant approximations.** Finding the "best" consonant approximation of a belief function is equivalent to approximating a belief measure with a necessity measure. B.f.s admit (among others) the following order relation

$$b \leq b' \equiv b(A) \leq b'(A) \;\; \forall A \subseteq \Theta \tag{1}$$

called *weak inclusion*. We can then define the *outer consonant approximations* [5] of a belief function $b$ as those co.b.f.s such that $co(A) \leq b(A)$ $\forall A \subseteq \Theta$ (or equivalently $pl_{co}(A) \geq pl_b(A)$ $\forall A$). With the purpose of finding outer approximations which are *minimal* with respect to the weak inclusion relation (1)) Dubois and Prade [5] introduced a family of outer consonant approximations obtained by considering all permutations $\rho$ of the elements $\{x_1, ..., x_n\}$ of the frame of discernment $\Theta$: $\{x_{\rho(1)}, ..., x_{\rho(n)}\}$. A family of nested sets can be then built $\{S_1^\rho = \{x_{\rho(1)}\}, S_2^\rho = \{x_{\rho(1)}, x_{\rho(2)}\}, ..., S_n^\rho = \{x_{\rho(1)}, ..., x_{\rho(n)}\}\}$ so that a new consonant belief function $co^\rho$ can be defined with b.p.a.

$$m_{co^\rho}(S_j^\rho) = \sum_{i:\min\{l: E_i \subseteq S_l^\rho\}=j} m_b(E_i). \tag{2}$$

$S_j^\rho$ is assigned the mass of the focal elements of $b$ included in $S_j^\rho$ but not in $S_{j-1}^\rho$.

## 3 The complex of consonant belief functions

A useful tool to represent uncertainty measures and discuss issues like the approximation problem is provided by convex geometry. Given a frame of discernment $\Theta$, a b.f. $b : 2^\Theta \rightarrow [0,1]$ is completely specified by its $N-2$ belief values $\{b(A), A \subseteq \Theta, A \neq \emptyset, \Theta\}$, $N \doteq 2^{|\Theta|}$, and can then be represented as a point of $\mathbb{R}^{N-2}$. The *belief space* associated with $\Theta$ is the set of points $\mathcal{B}$ of $\mathbb{R}^{N-1}$ which correspond to b.f.s. Let us call

$$b_A \doteq b \in \mathcal{B} \; s.t. \; m_b(A) = 1, \; m_b(B) = 0 \; \forall B \neq A \tag{3}$$

the unique b.f. assigning all the mass to a single subset $A$ of $\Theta$ ($A$-th *categorical* belief function). It can be proven that [2] the belief space $\mathcal{B}$ is the convex closure of all the categorical belief functions (3), $\mathcal{B} = Cl(b_A, \; \emptyset \subsetneq A \subseteq \Theta)$ where $Cl$ denotes the convex closure operator: $Cl(b_1, ..., b_k) = \{b \in \mathcal{B} : b = \alpha_1 b_1 + \cdots + \alpha_k b_k, \sum_i \alpha_i = 1, \alpha_i \geq 0 \; \forall i\}$.
More precisely $\mathcal{B}$ is an $N-2$-dimensional *simplex*, i.e. the convex closure of $N-1$ (affinely independent[1]) points of the Euclidean space $\mathbb{R}^{N-1}$. The *faces* of a simplex are all the simplices generated by a subset of its vertices. Each belief function $b \in \mathcal{B}$ can be written as a convex sum as $b = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) b_A$. Similarly the set of all Bayesian b.f.s is $\mathcal{P} = Cl(b_x, x \in \Theta)$.

**Binary example**. As an example consider a frame of discernment containing only two elements, $\Theta_2 = \{x, y\}$. Each b.f. $b : 2^{\Theta_2} \rightarrow [0,1]$ is determined by its belief values $b(x), b(y)$, as $b(\Theta) = 1$ and $b(\emptyset) = 0 \; \forall b$. We can then collect them in a vector of $\mathbb{R}^{N-2} = \mathbb{R}^2$:

$$[b(x) = m_b(x), b(y) = m_b(y)]' \in \mathbb{R}^2. \tag{4}$$

Since $m_b(x) \geq 0$, $m_b(y) \geq 0$, and $m_b(x) + m_b(y) \leq 1$ the set $\mathcal{B}_2$ of all the possible b.f.s on $\Theta_2$ is the triangle of Figure 1, whose vertices are the points $b_\Theta = [0,0]'$, $b_x = [1,0]'$, and $b_y = [0,1]'$. The region $\mathcal{P}_2$ of all Bayesian b.f.s on $\Theta_2$ is in this case the line segment $Cl(b_x, b_y)$. On the other side, consonant belief functions can have as chain of focal elements either $\{\{x\}, \Theta_2\}$ or $\{\{y\}, \Theta_2\}$. As a consequence the region $\mathcal{CO}_2$ of all co.b.f.s is the union of two segments: $\mathcal{CO}_2 = \mathcal{CO}_x \cup \mathcal{CO}_y = Cl(b_\Theta, b_x) \cup Cl(b_\Theta, b_y)$.

**The consonant simplicial complex**. The geometry of $\mathcal{CO}$ can be described in terms of a concept of convex geometry derived from that of simplex [6].

**Definition 1.** *A* simplicial complex *is a collection $\Sigma$ of simplices such that*

1. *if a simplex belongs to $\Sigma$, then all its faces are in $\Sigma$;*
2. *the intersection of two simplices is a face of both.*

Let us consider for instance two triangles on the plane (2-dimensional simplices). Roughly speaking, the second condition says that the intersection of those triangles cannot contain points of their interiors (Figure 2 left). It cannot also be

[1] An *affine combination* of $k$ points $v_1, ..., v_k \in \mathbb{R}^m$ is a sum $\alpha_1 v_1 + \cdots + \alpha_k v_k$ whose coefficients sum to one: $\sum_i \alpha_i = 1$. The affine subspace generated by the points $v_1, ..., v_k \in \mathbb{R}^m$ is the set $\{v \in \mathbb{R}^m : v = \alpha_1 v_1 + \cdots + \alpha_k v_k, \sum_i \alpha_i = 1\}$. If $v_1, ..., v_k$ generate an affine space of dimension $k$ they are said to be *affinely independent*.

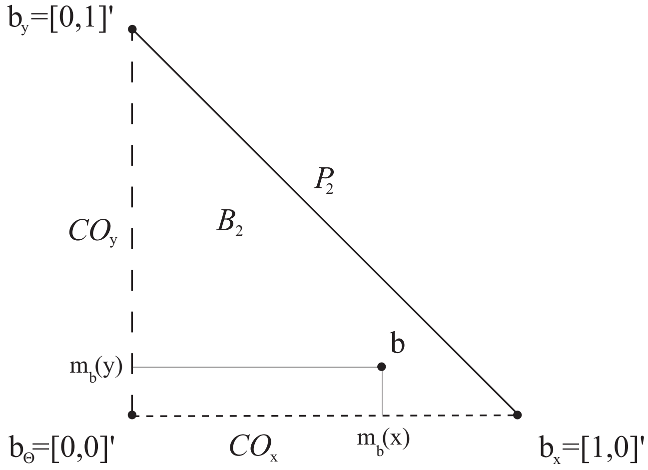


**Fig. 1.** The belief space $\mathcal{B}$ for a binary frame is a triangle in $\mathbb{R}^2$ whose vertices are the categorical belief functions $b_x$, $b_y$, $b_\Theta$ focused on $\{x\}, \{y\}$ and $\Theta$, respectively. The probability region is the segment $Cl(b_x, b_y)$. Consonant belief functions are constrained to belong to the union of the two segments $\mathcal{CO}_x = Cl(b_\Theta, b_x)$ and $\mathcal{CO}_y = Cl(b_\Theta, b_y)$.

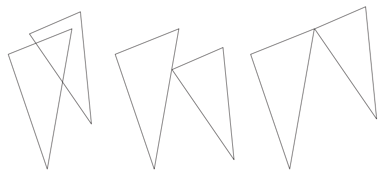

**Fig. 2.** Constraints on the intersection of simplices in a complex. Only the right-hand pair of triangles meets condition (2) of the definition of simplicial complex.

any subset of their borders (middle), but has to be a face (right, in this case a single vertex). It can be shown that [2]

**Proposition 1.** *$\mathcal{CO}$ is a simplicial complex included in the belief space $\mathcal{B}$.*

$\mathcal{CO}$ is the union of a collection of $\prod_{k=1}^{n} \binom{k}{1} = n!$ simplices, each associated with a maximal chain $\mathcal{C} = \{A_1 \subset \cdots \subset A_n = \Theta\}$ of $2^\Theta$:

$$\mathcal{CO} = \bigcup_{\mathcal{C}=\{A_1\subset\cdots\subset A_n\}} Cl(b_{A_1}, \cdots, b_{A_n}).$$

## 4 Outer approximations in the binary case

We can then study the geometry of the set $O[b]$ of all outer consonant approximations of a belief function $b$. In the binary case the latter is depicted in Figure 3 (dashed lines), as the intersection of the region of the points $b'$ with $b'(A) \leq b(A)$ $\forall A \subset \Theta$, and the complex $\mathcal{CO} = \mathcal{CO}_x \cup \mathcal{CO}_y$ of consonant b.f.s. Among them,

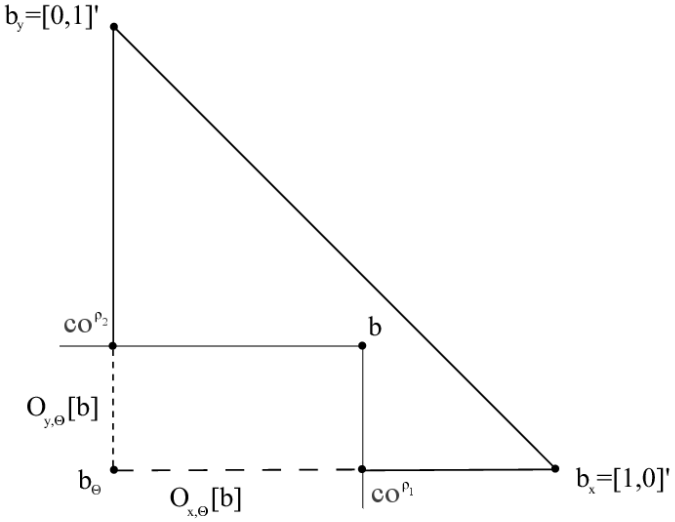


**Fig. 3.** Geometry of outer consonant approximations of a belief function $b \in \mathcal{B}_2$.

the co.b.f.s generated by the 2 possible permutations $\rho_1 = (x, y)$, $\rho_2 = (y, x)$ of elements of $\Theta_2$ as in (2) correspond to the points $co^{\rho_1}$, $co^{\rho_2}$ in Figure 3.

Let us denote by $O_{\mathcal{C}}[b]$ the intersection of the set $O[b]$ of all outer consonant approximations with the component $\mathcal{CO}_{\mathcal{C}}$ of the consonant complex, with $\mathcal{C}$ a maximal chain of $2^{\Theta}$. We can notice a number of interesting facts.
For each maximal chain $\mathcal{C}$:

1. $O_{\mathcal{C}}[b]$ is convex (in the binary case $\mathcal{C} = \{x, \Theta\}$ or $\{y, \Theta\}$);
2. $O_{\mathcal{C}}[b]$ is in fact a *polytope*, i.e. the convex closure of a number of vertices: in particular a segment in the binary case ($O_{x,\Theta}[b]$ or $O_{y,\Theta}[b]$);
3. the maximal (with respect to (1)) outer approximation of $b$ is one of the vertices of this polytope $O_{\mathcal{C}}[b]$, the one ($co^{\rho}$, Equation (2)) associated with the permutation $\rho$ of singletons which generates the chain.

In the binary case there are just two such permutations, $\rho_1 = \{x, y\}$ and $\rho_2 = \{y, x\}$, which generate respectively the chains $\{x, \Theta\}$ and $\{y, \Theta\}$.
We will prove that all those properties indeed hold in the general case.

## 5 Polytopes of outer consonant approximations

We first need a preliminary result on the basic probability assignment of consonant belief functions weakly included in $b$ [3].

**Weak inclusion and mass re-assignment.**

**Lemma 1.** *Consider a belief function b with basic probability assignment $m_b$. A consonant belief function co is weakly included in b, for all $A \subseteq \Theta$ $co(A) \leq b(A)$, if and only if there is a choice of coefficients $\{\alpha_A^B, B \subseteq \Theta, A \supseteq B\}$ with*

$$0 \leq \alpha_A^B \leq 1 \quad \forall B \subseteq \Theta, \forall A \supseteq B; \qquad \sum_{A \supseteq B} \alpha_A^B = 1 \quad \forall B \subseteq \Theta \tag{5}$$

*such that co has basic probability assignment*

$$m_{co}(A) = \sum_{B \subseteq A} \alpha_A^B m_b(B). \tag{6}$$

Lemma 1 states that the b.p.a. of any outer consonant approximation of $b$ is obtained by *re-assigning the mass of each f.e. $A$ of $b$ to some $B \supseteq A$*. We will extensively use this result in the following.

**Vertices of the polytopes.** Given a consonant belief function $co$ weakly included in $b$, its focal elements will form a chain $\mathcal{C} = \{B_1, ..., B_n\}$ ($|B_i| = i$) associated with a specific maximal simplex of $\mathcal{CO}$. According to Lemma 1 the mass of each focal element $A$ of $b$ can be re-assigned to some of the events of the chain $B_1, ..., B_n$ which contain $A$ in order to obtain $co$.
It is therefore natural to conjecture that, for each maximal simplex $\mathcal{CO}_\mathcal{C}$ of $\mathcal{CO}$ associated with a maximal chain $\mathcal{C}$, $O_\mathcal{C}[b]$ is the convex closure of the co.b.f.s $o^{\boldsymbol{B}}[b]$ with b.p.a.

$$m_{o^{\boldsymbol{B}}[b]}(B_i) = \sum_{A \subseteq \Theta : \boldsymbol{B}(A) = B_i} m_b(A) \tag{7}$$

each of them associated with an "assignment function"

$$\begin{array}{rl} \boldsymbol{B} : 2^\Theta & \rightarrow \mathcal{C} \\ A & \mapsto \boldsymbol{B}(A) \supseteq A \end{array} \tag{8}$$

which maps each event $A$ to one of the events of the chain $\mathcal{C} = \{B_1 \subset ... \subset B_n\}$ which contains $A$. As a matter of fact:

**Theorem 1.** *For each simplicial component $\mathcal{CO}_\mathcal{C}$ of the consonant space associated with any maximal chain of focal elements $\mathcal{C} = \{B_1, ..., B_n\}$ the set of outer consonant approximation of any b.f. b is the convex closure*

$$O_\mathcal{C}[b] = Cl(o^{\boldsymbol{B}}[b], \forall \boldsymbol{B})$$

*of the co.b.f.s (7) indexed by all admissible assignment functions (8).*

In other words, $O_\mathcal{C}[b]$ is a *polytope*, the convex closure of a number of b.f.s whose number is equal to the number of assignment functions (8). Each $\boldsymbol{B}$ is characterized by assigning each event $A$ to an element $B_i \supseteq A$ of the chain $\mathcal{C}$.

As we will see in the following ternary example the points (7) are not guaranteed to be all proper vertices of the polytope $O_\mathcal{C}[b]$. Some of them can be obtained as a convex combination of the others, i.e. they may lie on a side of the polytope.

**Maximal outer consonant approximations.** We can prove instead that the outer approximation (2) obtained by permuting the singletons of $\Theta$ as in Section 2 is not only a pseudo-vertex of $O_{\mathcal{C}}[b]$, but it is an actual vertex, i.e. it *cannot* be obtained as a convex combination of the others. More precisely, all possible permutations of elements of $\Theta$ generate exactly $n!$ different outer approximations of $b$, each of which lies on a single simplicial component of the consonant complex. Each such permutation $\rho$ generates a maximal chain $\mathcal{C}_\rho = \{S_1^\rho, ..., S_n^\rho\}$ of focal elements so that the corresponding b.f. will lie on $\mathcal{CO}_{\mathcal{C}_\rho}$.

**Theorem 2.** *The outer consonant approximation $co^\rho$ (2) generated by a permutation $\rho$ of the singletons is a vertex of $O_{\mathcal{C}_\rho}[b]$.*

We can prove that the maximal outer approximation is indeed the vertex $co^\rho$ associated with the corresponding permutation $\rho$ of the singletons which generates the maximal chain $\mathcal{C} = \mathcal{C}_\rho$ (as in the binary case of Section 4).
By definition (2) $co^\rho$ assigns the mass $m_b(A)$ of each focal element $A$ to the smallest element of the chain containing $A$. By Lemma 1 each outer consonant approximation of $b$ with chain $\mathcal{C}$, $co \in O_{\mathcal{C}_\rho}[b]$, is the result of re-distributing the mass of each focal element $A$ to all its supersets in the chain $\{B_i \supseteq A, B_i \in \mathcal{C}\}$. But then each such $co$ is weakly included in $co^\rho$ for its b.p.a. can be obtained by re-distributing the mass of the minimal superset $B_j$, where $j = \min\{i : B_i \subseteq A\}$, to all supersets of $A$.

**Corollary 1.** *The maximal outer consonant approximation with maximal chain $\mathcal{C}$ of a belief function $b$ is the vertex (2) of $O_{\mathcal{C}_\rho}[b]$ associated with the permutation $\rho$ of the singletons which generates $\mathcal{C} = \mathcal{C}_\rho$.*

**Example.** For a better understanding of the above results, let us consider as an example a belief function $b$ on a ternary frame $\Theta = \{x, y, z\}$ and study the polytope of outer consonant approximations with focal elements $\mathcal{C} = \{\{x\}, \{x, y\}, \{x, y, z\}\}$. According to Theorem 1 this is the convex closure of all assignment functions $\boldsymbol{B} : 2^\Theta \to \mathcal{C}$: there are $\prod_{k=1}^{3} k^{2^{3-k}} = 1^4 \cdot 2^2 \cdot 3^1 = 12$ such functions. We list them here according to the notation

$$\boldsymbol{B} = \boldsymbol{B}(\{x\}), \boldsymbol{B}(\{y\}), \boldsymbol{B}(\{z\}), \boldsymbol{B}(\{x, y\}), \boldsymbol{B}(\{x, z\}), \boldsymbol{B}(\{y, z\}), \boldsymbol{B}(\{x, y, z\}),$$

i.e.,

$$\begin{array}{llllll}
\boldsymbol{B}_1 = \{x\}, & \{x,y\}, \Theta, \{x,y\}, \Theta, \Theta, \Theta; & \boldsymbol{B}_7 = \{x,y\}, \Theta, & \Theta, \{x,y\}, \Theta, \Theta, \Theta; \\
\boldsymbol{B}_2 = \{x\}, & \{x,y\}, \Theta, \Theta, \quad \Theta, \Theta, \Theta; & \boldsymbol{B}_8 = \{x,y\}, \Theta, & \Theta, \Theta, \quad \Theta, \Theta, \Theta; \\
\boldsymbol{B}_3 = \{x\}, & \Theta, \quad \Theta, \{x,y\}, \Theta, \Theta, \Theta; & \boldsymbol{B}_9 = \Theta, & \{x,y\}, \Theta, \{x,y\}, \Theta, \Theta, \Theta; \\
\boldsymbol{B}_4 = \{x\}, & \Theta, \quad \Theta, \Theta, \quad \Theta, \Theta, \Theta; & \boldsymbol{B}_{10} = \Theta, & \{x,y\}, \Theta, \Theta, \quad \Theta, \Theta, \Theta; \\
\boldsymbol{B}_5 = \{x,y\}, & \{x,y\}, \Theta, \{x,y\}, \Theta, \Theta, \Theta; & \boldsymbol{B}_{11} = \Theta, & \Theta, \quad \Theta, \{x,y\}, \Theta, \Theta, \Theta; \\
\boldsymbol{B}_6 = \{x,y\}, & \{x,y\}, \Theta, \Theta, \quad \Theta, \Theta, \Theta; & \boldsymbol{B}_{12} = \Theta, & \Theta, \quad \Theta, \Theta, \quad \Theta, \Theta, \Theta.
\end{array}$$

They correspond to the following co.b.f.s with b.p.a. $[m(\{x\}), m(\{x,y\}), m(\Theta)]'$:

$$
\begin{array}{lllll}
o^{\boldsymbol{B}_1} & = [m_b(x), & m_b(y)+m_b(x,y), & 1-b(x,y) & ]'; \\
o^{\boldsymbol{B}_2} & = [m_b(x), & m_b(y), & 1-m_b(x)-m_b(y) & ]'; \\
o^{\boldsymbol{B}_3} & = [m_b(x), & m_b(x,y), & 1-m_b(x)-m_b(x,y) & ]'; \\
o^{\boldsymbol{B}_4} & = [m_b(x), & 0, & 1-m_b(x) & ]'; \\
o^{\boldsymbol{B}_5} & = [0, & b(x,y), & 1-b(x,y) & ]'; \\
o^{\boldsymbol{B}_6} & = [0, & m_b(x)+m_b(y), & 1-m_b(x)-m_b(y) & ]'; \\
o^{\boldsymbol{B}_7} & = [0, & m_b(x)+m_b(x,y), & 1-m_b(x)-m_b(x,y) & ]'; \\
o^{\boldsymbol{B}_8} & = [0, & m_b(x), & 1-m_b(x) & ]'; \\
o^{\boldsymbol{B}_9} & = [0, & m_b(y)+m_b(x,y), & 1-m_b(y)-m_b(x,y) & ]'; \\
o^{\boldsymbol{B}_{10}} & = [0, & m_b(y), & 1-m_b(y) & ]'; \\
o^{\boldsymbol{B}_{11}} & = [0, & m_b(x,y), & 1-m_b(x,y) & ]'; \\
o^{\boldsymbol{B}_{12}} & = [0, & 0, & 1 & ]'.
\end{array}
\qquad (9)
$$

Figure 4-left shows the resulting polytope $O_{\mathcal{C}}[b]$ for a belief function $m_b(x) =$

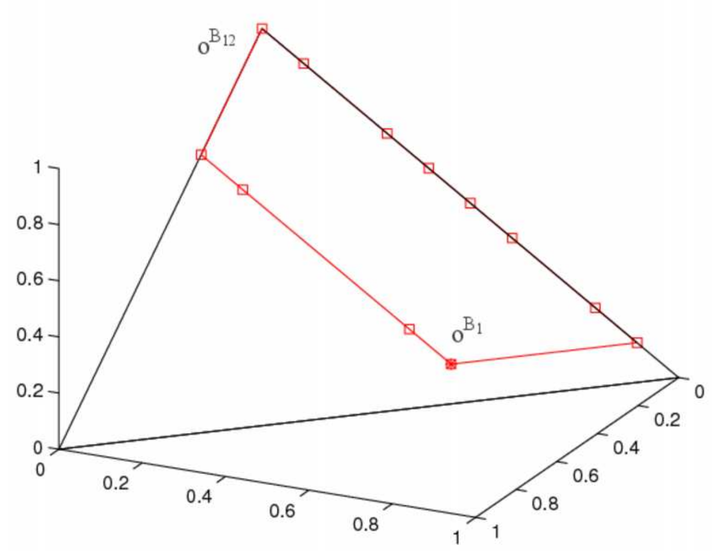


**Fig. 4.** Not all the points (7) associated with assignment functions are actual vertices of $O_{\mathcal{C}}[b]$. Here the polytope $O_{\mathcal{C}}[b]$ of outer consonant approximations for the belief function $m_b(x) = 0.3$, $m_b(y) = 0.5$, $m_b(\{x,y\}) = 0.1$, $m_b(\Theta) = 0.1$ defined on $\Theta = \{x,y,z\}$, with $\mathcal{C} = \{\{x\},\{x,y\},\Theta\}$ is plotted in red, together with all the 12 points (9) (red squares). Many of them lie on a side of the polytope. However, the point obtained by permutation of singletons (2) is an actual vertex (red star). The minimal and maximal outer approximations with respect to weak inclusion are $o^{\boldsymbol{B}_{12}}$ and $o^{\boldsymbol{B}_1}$, respectively.

0.3, $m_b(y) = 0.5$, $m_b(\{x,y\}) = 0.1$, $m_b(\Theta) = 0.1$, in the component $\mathcal{CO}_{\mathcal{C}} = Cl(b_x, b_{\{x,y\}}, b_\Theta)$ of the consonant complex (black triangle in the figure). The polytope $O_{\mathcal{C}}[b]$ is plotted in red, together with all the 12 points (9) (red squares). Many of them lie on some side of the polytope. However, the point obtained by permutation of singletons (2) is an actual vertex (red star): it is the first $o^{\boldsymbol{B}_1}$ of the list (9).

It is interesting to point out how the points (9) are ordered with respect to the weak inclusion relation (we just need to apply its definition, or the re-distribution property of Lemma 1). The result is summarized in the graph of Figure 5. We

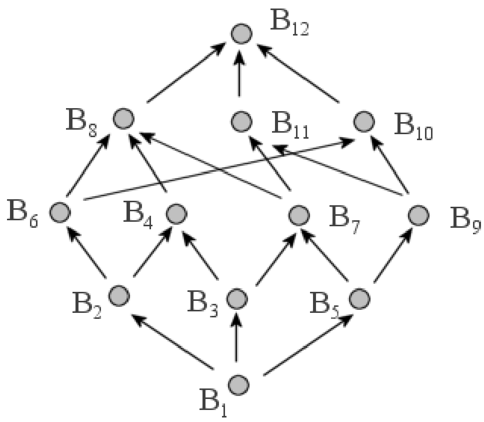


**Fig. 5.** Partial order of the points (9) with respect to the weak inclusion relation. For sake of simplicity we denote by $B_i$ the co.b.f. $o^{\boldsymbol{B}_i}$ associated with the assignment function $\boldsymbol{B}_i$. An arrow from $\boldsymbol{B}_i$ to $\boldsymbol{B}_j$ stands for $o^{\boldsymbol{B}_j} \leq o^{\boldsymbol{B}_i}$.

can appreciate that the vertex $o^{\boldsymbol{B}_1}$ generated by singleton permutation is indeed the maximal outer approximation of $b$, as stated by Corollary 1.

## 6 Conclusions

In this paper we studied the convex geometry of the consonant approximation problem, focusing in particular on the properties of outer consonant approximations. We showed that such approximations form a polytope in each maximal simplex of the complex $\mathcal{CO}$ of all consonant belief functions. We proved that for a given chain the maximal outer approximation is a vertex of the corresponding polytope and is generated by a permutation of the elements of the frame.
As they also live on simplicial complexes, natural extensions of this study to guaranteed possibility measures and consistent belief functions are in sight.

## Appendix

**Proof of Theorem 1.** We need to prove that:

1. each co.b.f. $co \in \mathcal{CO}_\mathcal{C}$ such that $co(A) \leq b(A)$ for all $A \subseteq \Theta$ can be written as a convex combination of the points (7): $co = \sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} o^{\boldsymbol{B}}[b]$, $\sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} = 1$, $\alpha_{\boldsymbol{B}} \geq 0 \ \forall \boldsymbol{B}$;
2. vice-versa, each convex combination of the $o^{\boldsymbol{B}}[b]$ satisfies $\sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} o^{\boldsymbol{B}}[b](A) \leq b(A)$ for all $A \subseteq \Theta$.

Let us consider (2) first. By definition of b.f. $o^{\boldsymbol{B}}[b](A) = \sum_{B\subseteq A, B\in\mathcal{C}} m_{o^{\boldsymbol{B}}[b]}(B)$ where $m_{o^{\boldsymbol{B}}[b]}(B) = \sum_{X\subseteq B:\boldsymbol{B}(X)=B} m_b(X)$ so that

$$o^{\boldsymbol{B}}[b](A) = \sum_{B\subseteq A, B\in\mathcal{C}} \ \sum_{X\subseteq B:\boldsymbol{B}(X)=B} m_b(X) = \sum_{X\subseteq B_i:\boldsymbol{B}(X)=B_j, j\leq i} m_b(X) \qquad (10)$$

where $B_i$ is the largest element of the chain $\mathcal{C}$ included in $A$. As $B_i \subseteq A$ (10) is obviously not larger than $\sum_{B\subseteq A} m_b(B) = b(A)$, so that $o^{\boldsymbol{B}}[b](A) \leq b(A)$ for all $A$. Hence $\forall A \subseteq \Theta$

$$\sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} o^{\boldsymbol{B}}[b](A) \leq \sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} b(A) = b(A) \sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} = b(A).$$

Let us prove point (1). According to Lemma 1, if $\forall A \subseteq \Theta$ $co(A) \leq b(A)$ then the mass $m_{co}(B_i)$ of each event $B_i$ of the chain is

$$m_{co}(B_i) = \sum_{A\subseteq B_i} m_b(A) \alpha^A_{B_i}. \qquad (11)$$

To prove (1) we then need to write (11) as a convex combination of the $m_{o^{\boldsymbol{B}}[b]}(B_i)$, i.e.

$$\sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} o^{\boldsymbol{B}}[b](B_i) = \sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} \sum_{X\subseteq B_i:\boldsymbol{B}(X)=B_i} m_b(X) = \sum_{X\subseteq B_i} m_b(X) \sum_{\boldsymbol{B}(X)=B_i} \alpha_{\boldsymbol{B}}.$$

In other words we need to show that the system of equations

$$\left\{ \alpha^A_{B_i} = \sum_{\boldsymbol{B}(A)=B_i} \alpha_{\boldsymbol{B}} \qquad \forall i = 1, ..., n; \quad \forall A \subseteq B_i \right. \qquad (12)$$

has at least one solution $\{\alpha_{\boldsymbol{B}}\}$ such that $\sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} = 1$ and $\forall \boldsymbol{B}$ $\alpha_{\boldsymbol{B}} \geq 0$. The normalization constraint is in fact trivially satisfied as from (12) it follows that

$$\sum_{B_i\supseteq A} \alpha^A_{B_i} = 1 = \sum_{B_i\supseteq A} \ \sum_{\boldsymbol{B}(A)=B_i} \alpha_{\boldsymbol{B}} = \sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}}$$

i.e. $\sum_{\boldsymbol{B}} \alpha_{\boldsymbol{B}} = 1$. Using the normalization constraint the system of equations (12) reduces to

$$\left\{ \alpha^A_{B_i} = \sum_{\boldsymbol{B}(A)=B_i} \alpha_{\boldsymbol{B}} \qquad \forall i = 1, ..., n-1; \quad \forall A \subseteq B_i. \right. \qquad (13)$$

We can show that each equation in the reduced system (13) involves at least one variable $\alpha_{\boldsymbol{B}}$ which is not present in any other equation. Formally, the set of assignment functions which meet the constraint of equation $A, B_i$ but not all others is not empty:

$$\left\{ \boldsymbol{B} : (\boldsymbol{B}(A) = B_i) \bigwedge_{\forall j=1,...,n-1; j \neq i} (\boldsymbol{B}(A) \neq B_j) \bigwedge_{\forall A' \neq A; \forall j=1,...,n-1} (\boldsymbol{B}(A') \neq B_j) \right\} \neq \emptyset. \tag{14}$$

But the assignment functions $\boldsymbol{B}$ such that $\boldsymbol{B}(A) = B_i$ and $\forall A' \neq A$ $\boldsymbol{B}(A') = \Theta$ all meet condition (14). Indeed they obviously meet $\boldsymbol{B}(A) \neq B_j$ for all $j \neq i$ while clearly for all $A' \subseteq \Theta$ $\boldsymbol{B}(A') = \Theta \neq B_j$, as $j < n$ so that $B_j \neq \Theta$.

A non-negative solution of (13) (and hence of (12)) can be obtained by setting for each equation one of such variables equal to the first member $\alpha^A_{B_i}$, and all the others to zero.

**Proof of Theorem 2.** The proof is divided in two parts.

1. We first need to find an assignment $\boldsymbol{B} : 2^\Theta \to \mathcal{C}_\rho$ which generates $co^\rho$. Each singleton $x_i$ is mapped by $\rho$ to the position $j$: $i = \rho(j)$. Then, given any event $A = \{x_{i_1}, ..., x_{i_m}\}$ its elements are mapped to the new positions $x_{j_{i_1}}, ..., x_{j_{i_m}}$, where $i_1 = \rho(j_{i_1}), ..., i_m = \rho(j_{i_m})$. But then the map

$$\boldsymbol{B}_\rho(A) = \boldsymbol{B}_\rho(\{x_{i_1}, ..., x_{i_m}\}) = S^\rho_j \doteq \{x_{\rho(1)}, ..., x_{\rho(j)}\}$$

where

$$j \doteq \max\{j_{i_1}, ..., j_{i_m}\}$$

maps each event $A$ to the smallest $S^\rho_i$ in the chain which contains $A$: $j = \min\{i : A \subseteq S^\rho_i\}$. Therefore it generates a co.b.f. with b.p.a. (2), i.e. $co^\rho$.

2. In order for $co^\rho$ to be an actual vertex, we need to ensure that it cannot be written as a convex combination of the other (pseudo) vertices $o^{\boldsymbol{B}}[b]$:

$$co^\rho = \sum_{\boldsymbol{B} \neq \boldsymbol{B}_\rho} \alpha_{\boldsymbol{B}} o^{\boldsymbol{B}}[b], \qquad \sum_{\boldsymbol{B} \neq \boldsymbol{B}_\rho} \alpha_{\boldsymbol{B}} = 1, \quad \forall \boldsymbol{B} \neq \boldsymbol{B}_\rho \ \alpha_{\boldsymbol{B}} \geq 0.$$

As $m_{o^{\boldsymbol{B}}}(B_i) = \sum_{A:\boldsymbol{B}(A)=B_i} m_b(A)$ the above condition reads as

$$\left\{ \sum_{A \subseteq B_i} m_b(A) \left( \sum_{\boldsymbol{B}:\boldsymbol{B}(A)=B_i} \alpha_{\boldsymbol{B}} \right) = \sum_{A \subseteq B_i : \boldsymbol{B}_\rho(A) = B_i} m_b(A) \qquad \forall B_i \in \mathcal{C}. \right.$$

Remembering that $\boldsymbol{B}_\rho(A) = B_i$ iff $A \subseteq B_i, \not\subset B_{i-1}$ we get

$$\left\{ \sum_{A \subseteq B_i} m_b(A) \left( \sum_{\boldsymbol{B}:\boldsymbol{B}(A)=B_i} \alpha_{\boldsymbol{B}} \right) = \sum_{A \subseteq B_i, \not\subset B_{i-1}} m_b(A) \qquad \forall B_i \in \mathcal{C}. \right.$$

For $i = 1$ the condition is $m_b(B_1) \left( \sum_{\boldsymbol{B}:\boldsymbol{B}(B_1)=B_1} \alpha_{\boldsymbol{B}} \right) = m_b(B_1)$ i.e.

$$\sum_{\boldsymbol{B}:\boldsymbol{B}(B_1)=B_1} \alpha_{\boldsymbol{B}} = 1, \qquad \sum_{\boldsymbol{B}:\boldsymbol{B}(B_1) \neq B_1} \alpha_{\boldsymbol{B}} = 0.$$

Replacing this condition in the second constraint $i = 2$ yields

$$m_b(B_2 \setminus B_1)\Bigg(\sum_{\substack{\boldsymbol{B}:\boldsymbol{B}(B_1)=B_1,\\ \boldsymbol{B}(B_2\setminus B_1)=B_2}} \alpha_{\boldsymbol{B}}\Bigg) + m_b(B_2)\Bigg(\sum_{\substack{\boldsymbol{B}:\boldsymbol{B}(B_1)=B_1,\\ \boldsymbol{B}(B_2)=B_2}} \alpha_{\boldsymbol{B}}\Bigg) = \\ = m_b(B_2 \setminus B_1) + m_b(B_2)$$

i.e.

$$m_b(B_2 \setminus B_1)\Bigg(\sum_{\substack{\boldsymbol{B}:\boldsymbol{B}(B_1)=B_1,\\ \boldsymbol{B}(B_2\setminus B_1)\neq B_2}} \alpha_{\boldsymbol{B}}\Bigg) + m_b(B_2)\Bigg(\sum_{\substack{\boldsymbol{B}:\boldsymbol{B}(B_1)=B_1,\\ \boldsymbol{B}(B_2)\neq B_2}} \alpha_{\boldsymbol{B}}\Bigg) = 0$$

which implies $\alpha_{\boldsymbol{B}} = 0$ for all the assignment functions $\boldsymbol{B}$ such that $\boldsymbol{B}(B_2 \setminus B_1) \neq B_2$ or $\boldsymbol{B}(B_2) \neq B_2$. The only non-zero coefficients can then be the $\alpha_{\boldsymbol{B}}$ s.t. $\boldsymbol{B}(B_1) = B_1$, $\boldsymbol{B}(B_2 \setminus B_1) = B_2$, $\boldsymbol{B}(B_2) = B_2$.
By induction you get that $\forall \boldsymbol{B} \neq \boldsymbol{B}_\rho$ we have $\alpha_{\boldsymbol{B}} = 0$ .

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