The intersection probability and its properties

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Outline

1. Approximation of probability intervals
2. Intersection probability
3. Credal geometry in the probability simplex
4. Affine family of transformations
5. Conclusions

Uncertainty measures as credal sets

- assume the possible answers to a problem Q form a finite set $\Theta = \{x_1, ..., x_n\}$ (frame)
- given some evidence we can describe our belief on the outcome of Q in different ways
- random sets, belief functions, probability intervals
- correspond to **credal sets**, i.e., convex sets of probability distributions

The probability transformation problem for BFs

- **probability transformation**: reducing a complex uncertainty measure to a single probability
- problem has been widely studied for BFs
- reasons: efficient evidence combination, decision making
- others have argued that Bayesian and belief calculi have the same expressive power
- proposals: pignistic function [Smets], relative plausibility [Voorbraak,Shenoy], relative belief [Cuzzolin], orthogonal projection [Cuzzolin], Sudano, etc

Probability intervals

- **interval probability system**: system of constraints on the probability values of a distribution $p : \Theta \rightarrow [0, 1]$ on Θ of the form:

$$(l, u) \doteq \Big\{ l(x) \leq p(x) \leq u(x), \forall x \in \Theta \Big\}$$

- determines a convex (credal) set of probabilities
- example: probability interval on a domain $\Theta = \{x, y, z\}$

$$0.2 \leq p(x) \leq 0.8, \quad 0.4 \leq p(y) \leq 1, \quad 0 \leq p(x) \leq 0.4.$$

Goal: transformation of prob intervals

- goal: sensible **probability transformation of probability intervals**
- need for sensible rationality principle (as in the pignistic case)
- proposal: **intersection probability**, originally found in the geometric approach to BFs [Cuzzolin IEEE SMC-B'07]
- it turns out to be a transformation of prob intervals, obeying a simple rationality axiom

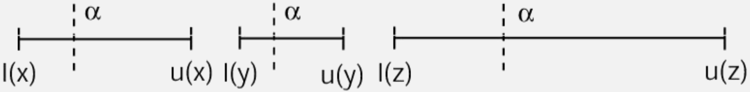
Rationale

of the intersection probability

- all the intervals $[l(x), u(x)]$, $x \in \Theta$ have the same relevance in a set of constraints
- reasonable request: the desired probability behave **homogeneously in each element** x of the frame Θ

$$p(x) = l(x) + \alpha(u(x) - l(x)) \qquad \forall x \in \Theta$$

for some $\alpha \in [0, 1]$



Intersection probability

analytic form

- unique solution: **intersection probability**

$$p[(l,u)](x) = \beta[(l,u)] \cdot u(x) + (1 - \beta[(l,u)]) \cdot l(x)$$

where

$$\beta[(l,u)] = \frac{1 - \sum_{x\in\Theta} l(x)}{\sum_{x\in\Theta} \big(u(x) - l(x)\big)}$$

Relative uncertainty of singletons

- most interesting interpretation of $p[(l,u)]$ comes from its alternative form

$$p[(l,u)](x) = l(x) + \left(1 - \sum_x l(x)\right) R[(l,u)](x)$$

where

$$R[(l,u)](x) \doteq \frac{u(x) - l(x)}{\sum_y (u(y) - l(y))} = \frac{\Delta(x)}{\sum_y \Delta(y)}$$

- $\Delta(x)$ measures the size of the probability interval on x
- $R(x)$ measures how much the width of the interval for x "weights" on the total width of the set of probability intervals
- **relative uncertainty of singletons**

Example

- same set of intervals as before

$$0.2 \leq p(x) \leq 0.8, \quad 0.4 \leq p(y) \leq 1, \quad 0 \leq p(x) \leq 0.4$$

- common fraction of the intervals

$$\beta = \frac{1 - 0.2 - 0.4 - 0}{0.6 + 0.6 + 0.4} = \frac{0.4}{1.6} = \frac{1}{4}$$

- values of the intersection probability

$$p[(l,u)](x) = 0.2 + \frac{1}{4}0.6 = 0.35, \quad p[(l,u)](y) = 0.4 + \frac{1}{4}0.6 = 0.55,$$
$$p[(l,u)](z) = 0 + \frac{1}{4}0.4 = 0.1$$

Intersection probability for BFs

- each belief measure also determines a set of probability intervals
- $\rightarrow$ the intersection probability can be defined for BFs too
- intervals associated with a BF $b(A) = \sum_{B \subseteq A} m_b(B)$

$$(b, pl_b) \doteq \{ p \in \mathcal{P} : b(x) \leq p(x) \leq pl_b(x), \forall x \in \Theta \}$$

- intersection prob for BFs: replace

$$l(x) \rightarrow b(x), \quad u(x) \rightarrow pl_b(x)$$

in the expression for generic prob intervals

Credal interpretation of BFs

and pignistic transform

- each belief function determines an entire set of probabilities *consistent* with it

$$\mathcal{P}[b] \doteq \Big\{ p \in \mathcal{P} : b(A) \leq p(A) \leq pl_b(A) \;\; \forall A \subseteq \Theta \Big\}$$

- *different* from the set of probabilities determined by the probability interval (b, pl_b)
- it is a convex set (credal set), with barycenter

$$BetP[b](x) = \sum_{A \supseteq \{x\}} \frac{m_b(A)}{|A|}$$

- **similar credal interpretation for the intersection probability**?

Credal interpretation of interval probs

Upper and lower simplices

- consider the set of probabilities which meet the **lower** constraint *on singletons* $T^1[b]$:

$$T^1[b] \doteq \left\{ p \in \mathcal{P} : p(x) \geq b(x) \;\; \forall x \in \Theta \right\}$$

- and the set of probs meeting the **upper** constraint *on singletons*

$$T^{n-1}[b] \;\; \doteq \;\; \left\{ p \in \mathcal{P} : p(x) \leq pl_b(x) \;\; \forall x \in \Theta \right\}$$

- the pair $(T^1[b], T^{n-1}[b])$ is the **geometric counterpart of an interval probability** in the probability simplex

Upper and lower simplices

- the form two higher-dimensional triangles or **simplices**
- $T^1[b] = Cl(t^1_x[b], x \in \Theta)$ with vertices

$$t^1_x[b] = \sum_{y \neq x} m_b(y) b_y + \Big(1 - \sum_{y \neq x} m_b(y)\Big) b_x$$

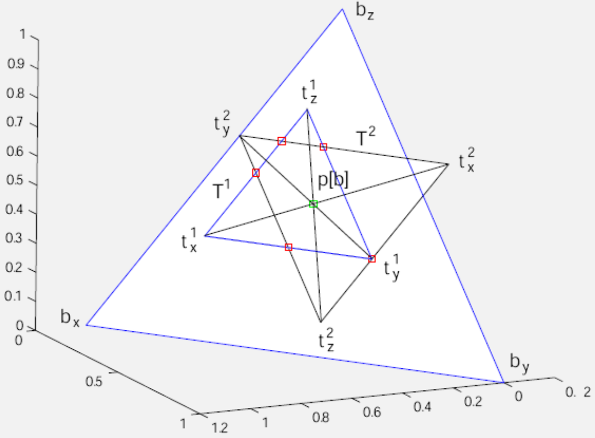
- $T^{n-1}[b] = Cl(t^{n-1}_x[b], x \in \Theta)$ with vertices

$$t^{n-1}_x[b] = \sum_{y \neq x} pl_b(y) b_y + \Big(1 - \sum_{y \neq x} pl_b(y)\Big) b_x$$

Ternary example

- example: the case of a BF on $\Theta = \{x, y, z\}$:

$$m_b(x) = 0.2, \quad m_b(y) = 0.1, \quad m_b(z) = 0.3,$$
$$m_b(\{x,y\}) = 0.1, \quad m_b(\{y,z\}) = 0.2, \quad m_b(\Theta) = 0.1$$



Focus of a pair of simplices

- $p[(l,u)]$ unique **intersection of the lines joining corresponding vertices** of upper and lower simplices
- can be formalized by the notion of "focus"
- **focus** of S, T: unique point with the **same affine coordinates** in both simplices

$$f(S,T) = \sum_{i=1}^{n} \alpha_i s_i = \sum_{j=1}^{n} \alpha_j t_j, \quad \sum_{i=1}^{n} \alpha_i = 1$$

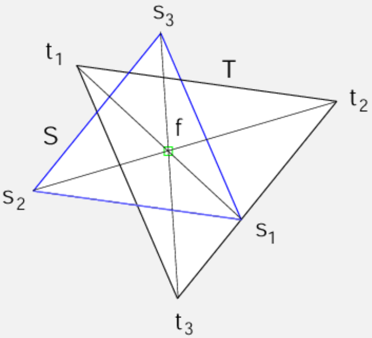
- does not always fall in their intersection $S \cap T$
- when it does, it coincides with intersection of lines passing through vertices

Intersection probability as focus

of upper and lower simplices

Theorem

The intersection probability is the focus of the pair of upper and lower simplices: $p[b] = f(T^{n-1}[b], T^1[b])$.



- interpretation: only probability meeting both constraints **in exactly the same way**
- **similar intepretation holds for relative belief and plausibility** [Cuzzolin, ISIPTA'09]

Affine family

of probability transformations

- as a transformation of BFs, intersection probability belongs to the **affine family**
- group of transformations which *commute with affine combination* (at least under certain conditions)

Theorem

Intersection probability $p[b]$ and affine combination commute:

$$p[\alpha_1 b_1 + \alpha_2 b_2] = \alpha_1 p[b_1] + \alpha_2 p[b_2]$$

for $\alpha_1 + \alpha_2 = 1$, if the relative uncertainty of the singletons is the same for both intervals: $R[b_1] = R[b_2]$

Comparison

with other members of the affine family

- other members of the affine family: *pignistic transform* $BetP[b]$ [Smets] and *orthogonal projection* $\pi[b]$ of b onto $\mathcal{P}$ [Cuzzolin, SMC-B'07]
- conditions under which they reduce to the same transformation?

Theorem

If a belief function b is such that its mass is ***equally distributed among focal elements of the same size***

$$m_b(A) = const \quad \forall A : |A| = k, \quad \forall k = 2, ..., n$$

then its pignistic transform, intersection probability, and orthogonal projection coincide: $BetP[b] = p[b] = \pi[b]$.

Conclusions

- probability transformation studied for BFs
- intersection probability: rational transformation *for sets of prob intervals*
- treats *all elements of the frame in the same way*
- this reflects in an elegant geometry in terms of focus of upper and lower simplices
- meets *upper and lower constraint in the same way*
- focus interpretation **extends to many other transformations**
- step towards a comprehensive classification of all probability transformations