Geometric conditioning of belief functions

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***Abstract*—In this paper we study the problem of conditioning a belief function b with respect to an event A by geometrically projecting such belief function onto the simplex associated with A in the simplex of all belief functions. Two different such simplices can be defined, as each belief function can be represented as the vector of its basic probability values or the vector of its belief values. We show here that defining geometric conditional b.f.s by minimizing L_p distances between b and the conditioning simplex in the mass space produces simple, elegant results with straightforward interpretations in terms of degrees of belief. This opens the way to a systematic exploration of geometric conditioning in the belief spacetoo, and the relationships of these results with classical approaches to the problem.**

Keywords: Belief functions, conditioning, geometric approach, L_p norms.

I. Introduction

Several theories and approaches to the issue of conditioning in the theory of belief functions [1], [2] have been proposed along the years [3]–[9]. In the original model in which belief functions are induced by multi-valued mappings of probability distributions, Dempster's conditioning can be judged inappropriate from a Bayesian point of view. Spies [10] defined conditional events as sets of equivalent events under conditioning. By applying multi-valued mapping to such events, conditional belief functions were introduced. An updating rule generalizing the total probability theorem was derived from them. Kyburg [11] analyzed the links between Dempster conditioning of belief functions and Bayesian conditioning of closed, convex sets of probabilities, of which belief functions are a special case. He arrived at the conclusion that the probability intervals generated by Dempster updating were included in those generated by Bayesian updating.

One way of dealing with such criticism is to abandon all notions of multivalued mapping to define belief directly in terms of basis belief assignments, as in Smets' transferable belief model [12]. The unnormalized conditional belief function $b_U(.|B)$ with b.b.a. $m_U(.|B)$[1]

$$m_U(.|B) = \begin{cases} \sum_{X \subseteq B^c} m(A \cup X) & if\ A \subseteq B \\ 0 & elsewhere \end{cases}$$

is the minimal commitment specialization of b such that $pl_b(B^c|B) = 0$ [13]. In [14], Xu and Smets used conditional belief functions to represent relations between variables in evidential networks, and presented a propagation algorithm for such networks. In [15], Smets pointed out the distinction between revision and focussing in the conditional process, and how they lead to unnormalized and geometric [16] conditioning $b_G(A|B) = \frac{b(A \cap B)}{b(B)}$, respectively. In these two scenarios he proposed generalizations of Jeffrey's rule of conditioning [17], [18] $P(A|P', \mathbb{B}) = \sum_{B \in \mathbb{B}} \frac{P(A \cap B)}{P(B)} P'(B)$ to the framework of belief functions.

Slobodova also conducted some early studies on the issue of conditioning. In particular, a multi-valued extension of conditional b.f.s was studied [19], and its properties examined. More recently, Tang and Zheng [20] also discussed the issue of conditioning in a multi-dimensional space. Klopotek and Wierzchon [21] provided a frequency-based interpretation for conditional belief functions.

Quite recently, Lehrer [22] proposed a geometric approach to determine the conditional expectation of non-additive probabilities. Such conditional expectation was then applied for updating, whenever information became available, and for introducing a notion of independence. Early attempts of studying conditioning in a geometric framework appear in [23], where the simplicial geometry of the set $\langle b \rangle$ of all belief functions obtained by Dempster combination with a given b.f. b, or *conditional subspace*, was described.

In this paper we propose instead of defining the notion itself of conditioning by geometric methods. The idea is simple: as the collection of events $\{B \subseteq A\}$ included in a given conditioning event A determine a simplex in the space of belief functions, conditional belief functions can be defined geometrically by minimizing a certain distance between the original b.f. b and the conditioning simplex. Such geometric conditioning can take place in two different spaces $\mathcal{M}$ and $\mathcal{B}$, according to whether we represent belief functions are vectors of mass values or belief values. We show here that defining geometric conditional b.f.s by minimizing L_p distances between b and the conditioning simplex in the mass space $\mathcal{M}$ produces simple, elegant results with straightforward interpretations in terms of degrees of belief. This opens the way to a systematic exploration of geometric conditioning in the belief space $\mathcal{B}$ too, and the relationships of these results with classical approaches to the problem.

We first briefly recall in Section II the geometric approach to belief functions. In particular, we show how each b.f. corresponds to a both the vector of its belief values in the belief space $\mathcal{B}$, and the vector of its mass values in the mass space $\mathcal{M}$. In this paper we consider the latter, and propose to

[1] Author's notation.

measure distances there by L_p norm. Therefore, in Sections III, IV and V we prove the analytical forms of the L_1, L_2 and L_∞ conditional belief functions in $\mathcal{M}$, and discuss their interpretation in terms of degrees of belief.

We conclude by outlining the future directions of this geometric approach to conditioning.

II. Geometric conditional belief functions

A. Belief functions as vectors

As belief functions $b : 2^\Theta \to [0, 1]$, $b(A) = \sum_{B \subseteq A} m_b(B)$, are set functions defined on a the power set 2^Θ of a finite space Θ, they are obviously completely defined by the associated set of $2^{|\Theta|} - 2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$ (since $b(\emptyset) = 0$, $b(\Theta) = 1$ for all b.f.s). They can therefore be represented as points of $\mathbb{R}^{N-2}, N = 2^{|\Theta|}$ [23]. The set $\mathcal{B}$ of points of $\mathbb{R}^{N-2}$ which correspond to a belief function is a simplex, namely: $\mathcal{B} = Cl(\vec{b}_A, \ \emptyset \subsetneq A \subseteq \Theta)$, where Cl denotes the convex closure operator and $\vec{b}_A$ is the vector associated with the categorical [12] belief function assigning all the mass to a single subset $A \subseteq \Theta$: $m_{b_A}(A) = 1$, $m_{b_A}(B) = 0$ for all $B \neq A$. The vector $\vec{b} \in \mathcal{B}$ that corresponds to a belief function b has in $\mathcal{B}$ coordinates: $\vec{b} = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) \vec{b}_A$.

In the same way, though, each belief function is uniquely associated with the related set of mass values $\{m(A), \emptyset \subsetneq A \subseteq \Theta\}$ (Θ this time included). It can therefore be seen also as a point of $\mathbb{R}^{N-1}$, the vector $\vec{m}$ of its $N - 1$ mass components:

$$\vec{m} = \sum_{\emptyset \subsetneq B \subseteq \Theta} m_b(B) \vec{m}_B. \tag{1}$$

Of course such vectors $\vec{m}$ will live in the subspace $\mathcal{M}$ of vectors whose components sum to 1.

B. Conditioning simplex and L_p norms

Similarly, the vector $\vec{a}$ associated with any belief function whose mass supports only focal elements $\{\emptyset \subsetneq B \subseteq A\}$ included in A can be decomposed as:

$$\vec{a} = \sum_{\emptyset \subsetneq B \subseteq A} m_a(B) \vec{m}_B. \tag{2}$$

The set of such vectors form a simplex $\mathcal{M}_A \doteq Cl(\vec{m}_B, \ \emptyset \subsetneq B \subseteq A)$. The same is true in the belief space, where each belief function $\vec{a}$ assigning mass to focal elements included in A can be decomposed as: $\vec{a} = \sum_{\emptyset \subsetneq B \subseteq A} a(B) \vec{b}_B$. These vectors live in a simplex $\mathcal{B}_A \doteq Cl(\vec{b}_B, \ \emptyset \subsetneq B \subseteq A)$. We call $\mathcal{M}_A$ and $\mathcal{B}_A$ the *conditioning simplex* in the mass and the belief space, respectively.

Given a belief function b, we call *geometric conditional belief function induced by a distance function* d in $\mathcal{M}$ ($\mathcal{B}$) the b.f.(s) $b_d(.|A)$ which minimize(s) the distance $d(b, \mathcal{M}_A)$ ($d(b, \mathcal{B}_A)$) between b and the conditioning simplex in $\mathcal{M}$ ($\mathcal{B}$). In this paper we consider mainly such geometric conditional b.f.s in the mass space $\mathcal{M}$.

We will consider as distance functions the three major L_p norms: $d = L_1$ (Section III); $d = L_2$ (Section IV); $d = L_\infty$ (Section V). For vectors $\vec{m}, \vec{m'} \in \mathcal{M}$ representing the b.p.a.s of two belief functions b, b', such norms read as:

$$\begin{aligned} \|\vec{m} - \vec{m'}\|_1 &\doteq \sum_{\emptyset \subsetneq B \subseteq \Theta} |m(B) - m'(B)|; \\ \|\vec{m} - \vec{m'}\|_2 &\doteq \sqrt{\sum_{\emptyset \subsetneq B \subseteq \Theta} (m(B) - m'(B))^2}; \\ \|\vec{m} - \vec{m'}\|_\infty &\doteq \max_{\emptyset \subsetneq B \subseteq \Theta} |m(B) - m'(B)|. \end{aligned} \tag{3}$$

III. Conditional belief functions by L_1 norm

Given any belief function b with basic probability assignment m_b collected in a vector $\vec{m}_b \in \mathcal{M}$, the L_1 conditional belief function $b_{L_1,\mathcal{M}}(.|A)$ is the unique b.f. whose basic probability assignment $m_{L_1,\mathcal{M}}(.|A)$ satisfies:

$$\vec{m}_{L_1,\mathcal{M}}(.|A) \doteq \arg\min_{\vec{a} \in \mathcal{M}_A} \|\vec{m}_b - \vec{a}\|_1. \tag{4}$$

Using the expression (3) of the L_1 norm in the mass space $\mathcal{M}$, (4) can be written as:

$$\arg\min_{\vec{a} \in \mathcal{M}_A} \|\vec{m}_b - \vec{a}\|_1 = \arg\min_{\vec{a} \in \mathcal{M}_A} \sum_{\emptyset \subsetneq B \subseteq \Theta} |m_b(B) - a(B)|.$$

A. A change of variables

By exploiting the fact that the candidate solution $\vec{a}$ is an element of $\mathcal{M}_A$ (Equation (2)) we can greatly simplify this expression. Namely,

$$\begin{aligned} \vec{m}_b - \vec{a} &= \sum_{\emptyset \subsetneq B \subseteq \Theta} m_b(B) \vec{m}_B - \sum_{\emptyset \subsetneq B \subseteq A} a(B) \vec{m}_B \\ &= \sum_{\emptyset \subsetneq B \subseteq A} (m_b(B) - a(B)) \vec{m}_B + \sum_{B \not\subseteq A} m_b(B) \vec{m}_B. \end{aligned}$$

The following change of variable

$$\beta(B) \doteq m_b(B) - a(B) \tag{5}$$

further yields:

$$\vec{m}_b - \vec{a} = \sum_{\emptyset \subsetneq B \subseteq A} \beta(B) \vec{m}_B + \sum_{B \not\subseteq A} m_b(B) \vec{m}_B. \tag{6}$$

We need to observe, though, that the variables $\{\beta(B), \emptyset \subsetneq B \subseteq A\}$ are not independent. Indeed,

$$\sum_{\emptyset \subsetneq B \subseteq A} \beta(B) = \sum_{\emptyset \subsetneq B \subseteq A} m_b(B) - \sum_{\emptyset \subsetneq B \subseteq A} a(B) = b(A) - 1$$

as $\sum_{\emptyset \subsetneq B \subseteq A} a(B) = 1$ by definition, since $\vec{a} \in \mathcal{M}_A$. Therefore in the optimization problem there are just $2^{|A|} - 2$ independent variables, while: $\beta(A) = b(A) - 1 - \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B)$. By replacing the above equality into (6) we finally get:

$$\begin{aligned} \vec{m}_b - \vec{a} = &\sum_{\emptyset \subsetneq B \subsetneq A} \beta(B) \vec{m}_B + \\ &+ \Big(b(A) - 1 - \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B) \Big) \vec{m}_A + \sum_{B \not\subseteq A} m_b(B) \vec{m}_B. \end{aligned} \tag{7}$$

B. L_1-conditional belief functions and dominating masses

In the L_1 case we get then

$$\|\vec{m}_b - \vec{a}\|_1 = \sum_{\emptyset \subsetneq B \subsetneq A} |\beta(B)| + \Big| b(A) - 1 - \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B) \Big|, \quad (8)$$

plus the constant $\sum_{B \not\subset A} |m_b(B)|$ which does not depend on β. This is a function of the form

$$\sum_i |x_i| + \Big| - \sum_i x_i - k \Big|, \quad k \geq 0 \quad (9)$$

which has an entire simplex of minima, namely: $x_i \leq 0 \ \forall i$, $\sum_i x_i \geq -k$. See Figure 1 for the case of two variables, x_1 and x_2 (corresponding to the L_1 conditioning problem on an event A of size $|A| = 2$). The minima of the L_1 norm (8) are

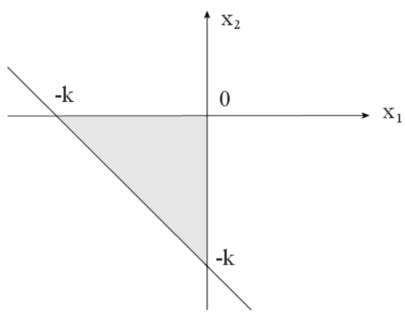

Figure 1. The minima of a function of the form (9) with two variables x_1, x_2 form the triangle $x_1 \leq 0$, $x_2 \leq 0$, $x_1 + x_2 \geq -k$ depicted here.

therefore given by the following system of constraints:

$$\begin{cases} \beta(B) \leq 0 & \forall \emptyset \subsetneq B \subsetneq A \\ \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B) \geq b(A) - 1. \end{cases} \quad (10)$$

In the original simplicial coordinates $\{a(B), \emptyset \subsetneq B \subseteq A\}$ of the candidate solution $\vec{a}$ in $\mathcal{M}_A$ this reads as:

$$\begin{cases} m_b(B) - a(B) \leq 0 & \forall \emptyset \subsetneq B \subsetneq A \\ \sum_{\emptyset \subsetneq B \subsetneq A} (m_b(B) - a(B)) \geq b(A) - 1 \end{cases}$$

i.e., $a(B) \geq m_b(B) \ \forall \emptyset \subsetneq B \subseteq A$.

Recall that the *core* $\mathcal{C}_b$ of a b.f. b is the union of its focal elements $B : m_b(B) \neq 0$.

Theorem 1: Given a belief function $b : 2^\Theta \to [0,1]$ and an arbitrary non-empty focal element $\emptyset \subsetneq A \subseteq \Theta$, the set of L_1 conditional belief functions $b_{L_1,\mathcal{M}}(.|A)$ with respect to A in $\mathcal{M}$ is the set of b.f.s with core in A such that their mass dominates that of b over all the subsets of A:

$$b_{L_1,\mathcal{M}}(.|A) = \{b' : \mathcal{C}_{b'} \subseteq A, m_{b'}(B) \geq m_b(B) \ \forall \emptyset \subsetneq B \subseteq A\}. \quad (11)$$

C. Simplex of L_1-conditional belief functions

As we may observe in Figure 1, the set of L_1 conditional belief function in $\mathcal{M}$ has geometrically the form of a simplex. It is easy to see that by Equation (10) the $2^{|A|} - 2$ vertices of such simplex are associated with the following solutions:

$$\begin{cases} \beta(X) &= 0 \ \forall \emptyset \subsetneq X \subsetneq A; \\ \beta(B) &= b(A) - 1, \ \beta(X) = 0 \ \forall \emptyset \subsetneq X \subsetneq A, X \neq B \\ & \quad \forall \emptyset \subsetneq B \subsetneq A. \end{cases}$$

Such solutions read in the $\{a(B)\}$ coordinates as the vectors $\vec{m}[b]|_{L_1}^B A = \vec{a} \in \mathcal{M}_A$ such that:

$$\begin{array}{l} a(B) = m_b(B) + 1 - b(A), \\ a(X) = m_b(X) \ \forall \emptyset \subsetneq X \subsetneq A, X \neq B \end{array} \quad (12)$$

for all $\emptyset \subsetneq B \subseteq A$ (A included).

Theorem 2: Given a b.f. $b : 2^\Theta \to [0,1]$ and an arbitrary non-empty focal element $\emptyset \subsetneq A \subseteq \Theta$, the set of L_1 conditional belief functions $b_{L_1,\mathcal{M}}(.|A)$ with respect to A in $\mathcal{M}$ is the simplex $\mathcal{M}_{L_1,A}[b] = Cl(\vec{m}[b]|_{L_1}^B A)$ with vertices (12).

It is important to notice that all the vertices of the L_1 conditional simplex fall inside $\mathcal{M}_A$ proper. In principle, some of them could have fallen in the linear space generated by $\mathcal{M}_A$ but outside the simplex $\mathcal{M}_A$, i.e., some of the solutions $a(B)$ could have been negative. This is not the case, support the validity of such an approach to conditioning based on geometric projections onto the appropriate simplices.

IV. Conditional belief functions by L_2 norm

We can proceed to find the L_2 conditional belief function(s) by using again the form (6) of the difference vector $\vec{m}_b - \vec{a}$, where again $\vec{a}$ is an arbitrary vector of the conditional simplex $\mathcal{M}_A$. In this case it is convenient to recall that the minimal L_2 distance between a point and a vector space is attained by the point of the vector space such that the difference vector is orthogonal to all the generators $\vec{g}_i$ of the vector space:

$$\arg\min_{\vec{q} \in V} \|\vec{p} - \vec{q}\|_2 = \hat{q} \in V : \langle \vec{p} - \vec{q}, \vec{g}_i \rangle = 0 \quad \forall i$$

whenever $\vec{p} \in \mathbb{R}^m$, $V = span(\vec{g}_i, i)$.

In our case $\vec{p} = \vec{m}_b$ is the original mass function, $\vec{q} = \vec{a}$ is an arbitrary point in $\mathcal{M}_A$, while the generators of $\mathcal{M}_A$ are all the vectors $\vec{g}_B = \vec{m}_B - \vec{m}_A$, $\forall \emptyset \subsetneq B \subsetneq A$. Such generators are vectors of the form

$$[0, \cdots, 0, 1, 0, \cdots, 0, -1, 0, \cdots, 0]'$$

with all zero entries but entry B (equal to 1) and entry A (equal to -1). Making use of Equation (6), the condition $\langle \vec{m}_b - \vec{a}, \vec{m}_B - \vec{m}_A \rangle = 0$ assumes then a very simple form

$$\beta(B) - b(A) + 1 + \sum_{\emptyset \subsetneq X \subsetneq A, X \neq B} \beta(X) = 0$$

for all possible generators of $\mathcal{M}_A$, i.e.:

$$2\beta(B) + \sum_{\emptyset \subsetneq X \subsetneq A, X \neq B} \beta(X) = b(A) - 1 \quad \forall \emptyset \subsetneq B \subsetneq A. \quad (13)$$

A. The unique solution of the L_2 problem

The system (13) is a linear system of $2^{|A|} - 2$ equations in $2^{|A|} - 2$ variables (the $\beta(X)$). It can therefore be written as

$$\mathcal{A}\vec{\beta} = (b(A) - 1)\vec{1},$$

where $\vec{1}$ is the vector of the appropriate size with all entries at 1. Its unique solution is trivially $\vec{\beta} = (b(A) - 1) \cdot \mathcal{A}^{-1}\vec{1}$. The matrix $\mathcal{A}$ is of the form

$$\mathcal{A} = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ & & \cdots & \\ 1 & 1 & \cdots & 2 \end{bmatrix}.$$

Simple linear algebra proves that the inverse of such matrix has the form

$$\mathcal{A}^{-1} = \frac{1}{d+1} \left[\begin{array}{cccc} d & -1 & \cdots & -1 \\ -1 & d & \cdots & -1 \\ & & \cdots & \\ -1 & -1 & \cdots & d \end{array} \right]$$

where d is the number of rows (or columns) of $\mathcal{A}$. It is easy to see that $\mathcal{A}^{-1}\vec{1} = \frac{1}{d+1}\vec{1}$, where in our case $d = 2^{|A|} - 2$. The solution of the system (13) is then given by

$$\vec{\beta} = \mathcal{A}^{-1}\vec{1}(b(A) - 1) = \frac{1}{2^{|A|} - 1}\vec{1}(b(A) - 1),$$

i.e., then $\arg\min_{\vec{\beta}} \|\vec{m}_b - \vec{a}\|_\infty$ is such that

$$\beta(B) = \frac{b(A) - 1}{2^{|A|} - 1} \quad \forall \emptyset \subsetneq B \subsetneq A. \tag{14}$$

In the $\{a(B)\}$ coordinates the solution reads as

$$a(B) = m_b(B) + \frac{1 - b(A)}{2^{|A|} - 1} \quad \forall \emptyset \subsetneq B \subseteq A, \tag{15}$$

A included.

B. *L_2 conditional b.f. as barycenter of dominating masses*

According to Equation (15) then, the L_2 conditional belief function is unique, and corresponds to the mass function which *redistributes in an equal, even way the mass the original belief function assign to focal elements not included in A to each and all the subsets of A.*

Theorem 3: Given a belief function $b : 2^\Theta \to [0, 1]$ and an arbitrary non-empty focal element $\emptyset \subsetneq A \subseteq \Theta$, the unique L_2 conditional belief functions $b_{L_2,\mathcal{M}}(.|A)$ with respect to A in $\mathcal{M}$ is the b.f. whose b.p.a. redistributes the mass $1 - b(A)$ to each focal element $B \subseteq A$ in an equal way:

$$m_{L_2,\mathcal{M}}(B|A) = m_b(B) + \frac{1}{2^{|A|} - 1} \sum_{B \not\subset A} m_b(B) \; \forall \emptyset \subsetneq B \subseteq A.$$

Besides, L_2 and L_1 conditional belief functions in $\mathcal{M}$ display a strong relationship, as:

Theorem 4: Given a belief function $b : 2^\Theta \to [0, 1]$ and an arbitrary non-empty focal element $\emptyset \subsetneq A \subseteq \Theta$, the L_2 conditional belief functions $b_{L_2,\mathcal{M}}(.|A)$ with respect to A in $\mathcal{M}$ is the center of mass of the simplex $\mathcal{M}_{L_1,A}[b]$ of L_1 conditional belief functions with respect to A in $\mathcal{M}$.

Proof: By definition the center of mass of $\mathcal{M}_{L_1,A}[b]$, whose vertices are given by (12), is the vector

$$\frac{1}{2^{|A|} - 1} \sum_{\emptyset \subsetneq B \subseteq A} \vec{m}[b]|^B_{L_1} A$$

whose entry B is given by

$$\frac{1}{2^{|A|} - 1} \Big[m_b(B)(2^{|A|} - 1) + (1 - b(A)) \Big]$$

i.e., (15). ■

V. CONDITIONAL BELIEF FUNCTIONS BY L_∞ NORM

Similarly, we can use Equation (7) to minimize the L_∞ distance between the original mass vector $\vec{m}_b$ and the conditioning subspace $\mathcal{M}_A$. Let us rewrite it for sake of readability:

$$\begin{aligned} \vec{m}_b - \vec{a} &= \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B)\vec{m}_B + \sum_{B \not\subset A} m_b(B)\vec{m}_B \\ &+ \Big(b(A) - 1 - \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B) \Big) \vec{m}_A. \end{aligned}$$

Its L_∞ norm reads as:

$$\begin{aligned} \|\vec{m}_b - \vec{a}\|_\infty &= \max \Big\{ |\beta(B)| \; \emptyset \subsetneq B \subsetneq A, \\ & |m_b(B)| \; B \not\subset A, \Big| b(A) - 1 - \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B) \Big| \Big\}. \end{aligned}$$

As $|b(A) - 1 - \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B)| = | - \sum_{B \not\subset A} m_b(B) - \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B)| = |\sum_{B \not\subset A} m_b(B) + \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B)|$, the above norms simplifies as:

$$\begin{aligned} \max \Big\{ & |\beta(B)| \; \emptyset \subsetneq B \subsetneq A, \max_{B \not\subset A}\{m_b(B)\}, \\ & \Big| \sum_{B \not\subset A} m_b(B) + \sum_{\emptyset \subsetneq B \subsetneq A} \beta(B) \Big| \Big\}. \end{aligned} \tag{16}$$

This is a function of the form

$$\begin{aligned} f(x_1, ..., x_n) &= \max \Big\{ |x_i| \; \forall i, |\sum_i x_i + k_1|, k_2 \Big\} \\ &= \max \Big\{ g(x_1, ..., x_n), k_2 \Big\} \end{aligned}$$

where $g(x_1, ..., x_n) = \max \Big\{ |x_i| \;\; \forall i, |\sum_i x_i + k_1| \Big\}$, with $k_1, k_2 \geq 0$, $k_2 \leq k_1$. The shape of a function of the form g is represented (in the case in which there are two variables, i.e., when $A = 2$) in Figure 2.

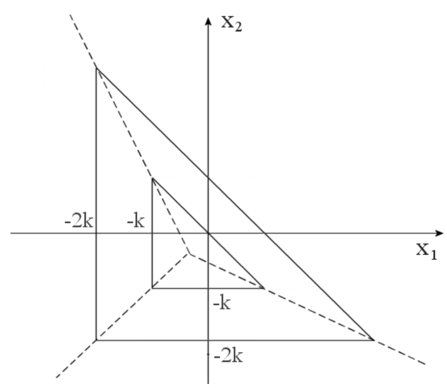

Figure 2. The level sets of functions of the form $g(x_1, x_2) = \max\{|x_1|, |x_2|, |x_1 + x_2 + k_1|\}$, $k_1 \geq 0$ are triangles with vertices $[-x, -x]'$, $[-x, +x]'$, $[x, -x]'$. The minimum of such a function g lies in $[x_1, x_2]' = [-k_1/3, -k_1/3]'$.

The minimum of such a function lies in $\frac{1}{n+1}[-k_1, ..., -k_1]'$, where n is the number of variables. The function there assumes value $\min g = \frac{k_1}{n+1}$. Its level sets have the form of simplices of the kind depicted in Figure 2 for the case of 2 variables.

Now, what is the minimum of $f(x_1, ..., x_n)$? Clearly, if $k_2 \leq \min g = \frac{k_1}{n+1}$ then f has also minimum in $\frac{1}{n+1}[-k_1, ..., -k_1]'$.
Immediately, since (16) is a function of the form g in $n = 2^{|A|} - 2$ variables (the $\beta(B)$) we obtain the following result.

Lemma 1: If $\max_{B \not\subset A}\{m_b(B)\} \leq \frac{1-b(A)}{2^{|A|}-1}$ then $\arg\min_{\vec{\beta}} \|\vec{m}_b - \vec{a}\|_\infty$ is such that

$$\beta(B) = \frac{b(A) - 1}{2^{|A|} - 1} \quad \forall \emptyset \subsetneq B \subsetneq A. \tag{17}$$

As an immediate consequence:

Theorem 5: If

$$\max_{B \not\subset A}\{m_b(B)\} \leq \frac{\sum_{B \not\subset A} m_b(B)}{2^{|A|} - 1}$$

then the L_∞ conditional belief function $b_{L_\infty,\mathcal{M}}(.|A)$ with respect to A in $\mathcal{M}$ is unique, and corresponds to the L_2 conditional belief function (15), and the barycenter of the polytope of L_1 conditional belief functions.

Proof: It suffices to notice that (17) coincides with (14). ∎

VI. Discussion and perspectives

A. Features of geometric conditional belief functions

From the analysis of geometric conditioning in the space of mass functions $\mathcal{M}$ a number of facts arise:

- geometric conditional b.f.s, even though obtained by minimizing purely geometric distances, possess very simple and elegant interpretations in terms of degrees of belief;
- while some of them correspond to pointwise conditioning, some others form whole polytopes of solutions whose vertices also have simple interpretations;
- conditional belief functions associated with the major L_1, L_2 and L_∞ norms are strictly related to each other;
- they are all characterized by the fact that, in the way they re-assign mass from focal elements $B \not\subset A$ not in A to focal elements in A, they do not distinguish between subsets which have non-empty intersection with A and those which have not.

The last point is quite interesting: mass geometric conditional b.f.s do not seem to care about the contribution focal elements make *to the plausibility* of the conditioning event A, but only to whether they contribute or not to the *degree of belief* of A. The reason is, roughly speaking, that in mass vectors $\vec{m}$ the mass of a given focal element appears only in the corresponding entry of $\vec{m}$. In opposition, belief vectors $\vec{b}$ are such that each entry $\vec{b}(B) = \sum_{X \subseteq B} m_b(X)$ of theirs contains information about the mass of all the subsets of B. As a result, it is to be expected that geometric conditioning *in the belief space* $\mathcal{B}$ will see the mass redistribution process function in a manner linked to the contribution of each focal element to the plausibility of the conditioning event A. Due to lack of space we will analyze and discuss the toy case of a ternary frame for future reference.

B. Geometric conditioning in the belief space: a first step

The problem of projecting a belief function b represented by the corresponding vector $\vec{b}$ of belief values onto a conditioning subspace $\mathcal{B}_A = Cl(b_B, \emptyset \subsetneq B \subseteq A)$ starts by explicitly writing the difference vector $\vec{b} - \vec{a}$ between $\vec{b}$ and an arbitrary point $\vec{a}$ of $\mathcal{B}_A$. The latter quite obviously reads as

$$\begin{aligned}\vec{b} - \vec{a} &= \sum_{\emptyset \subsetneq B \subseteq \Theta} m_b(B) b_B - \sum_{\emptyset \subsetneq B \subseteq A} a(B) b_B \\ &= \sum_{\emptyset \subsetneq B \subseteq A} (m_b(B) - a(B)) b_B + \sum_{B \not\subset A} m_b(B) b_B \\ &= \sum_{\emptyset \subsetneq B \subseteq A} \beta(B) b_B + \sum_{B \not\subset A} m_b(B) b_B.\end{aligned}$$

In the case of a size-3 frame $\Theta = \{x, y, z\}$, and a conditioning event of size 2 (for instance $A = \{x, y\}$) this particularizes as

$$\begin{aligned}\vec{b} - \vec{a} &= \beta(x) b_x + \beta(y) b_y + \beta(x,y) b_{\{x,y\}} \\ &\quad + m_b(z) b_z + m_b(x,z) b_{\{x,z\}} + m_b(y,z) b_{\{y,z\}},\end{aligned} \tag{18}$$

as $b_\Theta = \vec{0}$.

1) The L_2 case: To find the vector $\vec{a} \in \mathcal{B}_A$ which minimizes the L_2 distance from $\vec{b}$ (its L_2 conditional belief function $b_{L_2,\mathcal{B}}(.|A)$ with respect to A in $\mathcal{B}$) we need again to impose the conditions (remember Section IV):

$$\langle \vec{b} - \vec{a}, b_x - b_{\{x,y\}} \rangle = 0, \quad \langle \vec{b} - \vec{a}, b_y - b_{\{x,y\}} \rangle = 0$$

as $\mathcal{B}_{\{x,y\}} = Cl(b_x, b_y, b_{\{x,y\}})$ has two generators: $b_x - b_{\{x,y\}}$ and $b_y - b_{\{x,y\}}$.

Remembering that $\langle b_A, b_B \rangle = \langle b_{A \cup B}, b_{A \cup B} \rangle = 2^{|(A \cup B)^c|} - 1$, after simple maths we have the following linear system:

$$\begin{cases} 2\beta(x) + m_b(z) + m_b(x,z) = 0 \\ 2\beta(y) + m_b(z) + m_b(y,z) = 0 \end{cases} \tag{19}$$

whose solution is clearly

$$\beta(x) = -\frac{m_b(z) + m_b(x,z)}{2}, \quad \beta(y) = -\frac{m_b(z) + m_b(y,z)}{2}$$

which corresponds to

$$\begin{aligned}&a(x) = m_b(x) + \tfrac{m_b(z) + m_b(x,z)}{2}, \\ &a(y) = m_b(y) + \tfrac{m_b(z) + m_b(y,z)}{2}, \\ &a(x,y) = m_b(x,y) + m_b(\Theta) + \tfrac{m_b(x,z) + m_b(y,z)}{2}.\end{aligned}$$

At a first glance, each focal element $B \subseteq A$ seems to be assigned a fraction of the original mass $m_b(X)$ of all focal elements X of b such that $X \subseteq B \cup A^c$. This contribution seems proportional to the size of $X \cap A^c$, i.e., how much the focal element of b falls outside the conditioning event A.

Notice that Dempster's conditioning $b_\oplus(.|A) = b \oplus b_A$ yields in this case:

$$\begin{aligned}m_\oplus(x|A) &= m_{b \oplus b_A}(x) = \frac{m_b(x) + m_b(x,z)}{1 - m_b(z)}, \\ m_\oplus(y|A) &= m_{b \oplus b_A}(y) = \frac{m_b(y) + m_b(y,z)}{1 - m_b(z)}, \\ m_\oplus(x,y|A) &= m_{b \oplus b_A}(x,y) = \frac{m_b(x,y) + m_b(\Theta)}{1 - m_b(z)}.\end{aligned}$$

L_2 conditioning in the belief space differs from its "sister" operation in the mass space in that it makes use of the set-theoretic relations between focal elements, as Dempster's rule does. However, contrary to Dempster's conditioning it does not apply any normalization, as even subsets of A^c ($\{z\}$ in this case) contribute as addenda to the mass of the resulting conditional belief function.

2) The L_1 case: To discuss L_1 conditioning in the ternary belief space we need to write explicitly the vector (18). After recalling that vector $\vec{b}_A$ associated with a categorical b.f. b_A has as entries $\vec{b}_A(B) = 1$ if $B \supseteq A$, 0 otherwise, we get: $\vec{b} - \vec{a} =$

$$\begin{bmatrix} \beta(x), \beta(y), m_b(z), \beta(x) + \beta(y) + \beta(x,y), \\ \beta(x) + m_b(z) + m_b(x,z), \beta(y) + m_b(z) + m_b(x,z) \end{bmatrix},$$

so that $\|\vec{b} - \vec{a}\|_1 = |\beta(x)| + |\beta(y)| + |m_b(z)| + |\beta(x) + \beta(y) + \beta(x,y)| + |\beta(x) + m_b(z) + m_b(x,z)| + |\beta(y) + m_b(z) + m_b(x,z)|$, and we seek

$$\arg\min_{\beta(x),\beta(y)} \Big\{ \begin{array}{l} |\beta(x)| + |\beta(x) + m_b(z) + m_b(x,z)| + \\ +|\beta(y)| + |\beta(y) + m_b(z) + m_b(x,z)| \end{array} \Big\}.$$

The function to optimize here is the sum of two functions of the form $|x| + |x + k|$, which is minimized for $-k \leq x \leq 0$. In our case the solution is

$$\begin{array}{l} -(m_b(z) + m_b(x,z)) \leq \beta(x) \leq 0, \\ -(m_b(z) + m_b(y,z)) \leq \beta(y) \leq 0, \end{array}$$

i.e.: $b(x) \leq a(x) \leq b(x,z)$, $b(y) \leq a(y) \leq b(y,z)$.
The interpretation is simple:

Theorem 6: Given a b.f. $b : 2^{\{x,y,z\}} \rightarrow [0,1]$, the L_1 conditional belief functions $b_{L_1,\{x,y,z\}}(.|\{x,y\})$ with respect to $A = \{x,y\}$ in $\mathcal{B} = \{x,y,z\}$ are all those b.f.s with core included in A such that the conditional mass of $B \subsetneq A$ is between $b(B)$ and $b(B \cup A^c)$: $b(B) \leq m_{L_1,\mathcal{B}}(B|A) \leq b(B \cup A^c)$.
One could be tempted to conjecture that this simple and elegant behavior is indeed a general feature of L_1 conditioning on general belief spaces.

Another fact one can remark is that the barycenter of the L_1 solutions is $\beta(x) = -\frac{m_b(z)+m_b(x,z)}{2}$, $\beta(y) = -\frac{m_b(z)+m_b(y,z)}{2}$, i.e., the L_2 conditional b.f., just like in the case of geometric conditioning in the mass space $\mathcal{M}$. The same can be easily proved for all $A \subseteq \{x,y,z\}$.

Theorem 7: For every belief function $b : 2^{\{x,y,z\}} \rightarrow [0,1]$, the unique L_2 conditional b.f. $b_{L_1,\{x,y,z\}}(.|\{x,y\})$ with respect to $A \subseteq \{x,y,z\}$ in $\mathcal{B} = \{x,y,z\}$ is the barycenter of the polytope of L_1 conditional b.f.s with respect to A in $\mathcal{B}$.
We will work on the interesting conjectures arising from the analysis of the ternary case in the very near future.

VII. Conclusions

In this paper we showed how the notion of conditional belief function $b(.|A)$ can be introduced by geometric ways, by projecting any belief function onto the the simplicial subspace associated with the event A. The result will obviously depend on the choice of the vectorial representation for b, and of the distance function to minimize. We thoroughly analyzed the case of conditioning a b.b.a. vector by means of the norms L_1, L_2, and L_∞, and showed how the results have simple interpretations in terms of degrees of belief. A complete analysis of geometric conditioning in the belief space is the next obvious step. In the near future it will be interesting to understand how geometric conditional b.f.s related to classical approaches to conditioning, and whether the latter can themselves be seen as the result of minimizing some distance function in some geometric framework.

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