Consonant approximations in the belief space

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Abstract In this paper we solve the problem of approximating a belief measure with a necessity measure or "consonant belief function" by minimizing appropriate distances from the consonant complex in the space of all belief functions. Partial approximations are first sought in each simplicial component of the consonant complex, while global solutions are obtained from the set of partial ones. The L_1, L_2 and L_∞ consonant approximations in the belief space are here computed, discussed and interpreted as generalizations of the maximal outer consonant approximation. Results are also compared to other classical approximations in a ternary example.

1 Introduction

The theory of evidence [1] is a popular approach to uncertainty description in which probabilities are replaced by *belief functions* (b.f.s), functions $b : 2^\Theta \to [0,1]$ on the power set $2^\Theta = \{A \subseteq \Theta\}$ of the sample space Θ of the form $b(A) = \sum_{B \subseteq A} m_b(B)$, where $m_b : 2^\Theta \to [0,1]$ is a non-negative, normalized set function called "basic probability assignment" (b.p.a.) or "mass assignment", and $pl_b(A) \doteq 1 - b(A^c)$ is the *plausibility function* (pl.f.) associated with b. Belief functions assign values $b(A)$ between 0 and 1 to subsets of the sample space Θ rather than single elements. In opposition, possibility theory [2] studies *possibility measures*, i.e., functions $Pos : 2^\Theta \to [0,1]$ on the power set such that $Pos(\bigcup_i A_i) = \sup_i Pos(A_i)$ for any family of subsets $\{A_i | A_i \in 2^\Theta, i \in I\}$, where I is an arbitrary set index. Given a possibility measure Pos, the dual *necessity* measure is defined as $Nec(A) = 1 - Pos(A^c)$. Interestingly, necessity measures have counterparts in the theory of evidence in the form of *consonant* belief functions (co.b.f.s), i.e., b.f.s whose non-zero mass subsets $m_b(A) \neq 0$ or "focal elements" (f.e.s) are nested [1] and form a chain (totally ordered collection) of subsets $A_1 \subset \cdots \subset A_m$, $A_i \subseteq \Theta$, in which case $Pos(x) = pl_b(x)$.

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Approximating a b.f. with a necessity measure amounts therefore to mapping it to a consonant b.f. [3, 4, 5]. As possibilities are completely determined by their values on the singletons ($Pos(\{x\})$, $x \in \Theta$), they are less computationally expensive than b.f.s, making the approximation process interesting for many applications.

A geometric approach to consonant approximation. Dubois and Prade, in particular, have proposed the notion of "outer consonant approximations" [3] of belief functions. Their work has been later extended by Baroni [5] to capacities, while, in [6], the author has provided a description of the geometry of the set of outer consonant approximations. A different "isopignistic" approximation has been proposed as the unique consonant b.f. whose pignistic probability $BetP(x) = \sum_{A \supseteq x} m_b(A)$ is identical to that of the original b.f. b [8, 9, 10]. In more recent times, the opportunity of seeking probability or consonant approximations / transformations of belief functions by minimizing appropriate distance functions has been explored [11, 13]. Any dissimilarity measure could be in principle employed to define conditional b.f.s, or to approximate b.f.s by necessity or probability measures [15, 16, 17]. We focus here on classical L_p norms which have been successfully applied in the past [12, 14].

Contribution. The goal of this paper is to conduct an analytical study of all the consonant approximations induced by minimizing L_1, L_2 or L_∞ distances between the original belief function and the consonant region, in the vector space they form or *belief space*, as a stepping stone of a more extensive theoretical study of the nature of consonant approximations induced by geometric distance minimization.

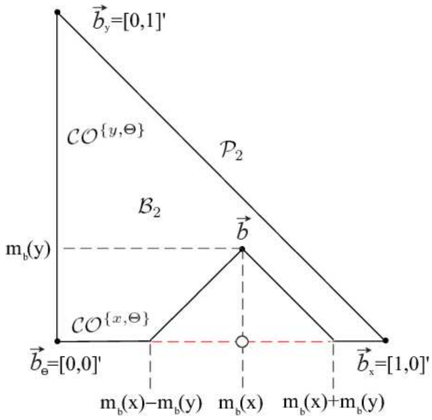
As it turns out, all partial L_p consonant approximations in $\mathscr{B}$ amount to picking different representatives from the n lists of belief values: $\mathscr{L}^i = \{b(A), A \supseteq A_i, A \not\supset A_{i+1}\}$ $\forall i = 1,...,n$, as they have mass $m'(A_i) = f(\mathscr{L}^i) - f(\mathscr{L}^{i-1})$, where f is a simple function of the belief values in the list, such as max, average, or median. Classical maximal outer and "contour-based" approximations can also be expressed in the same way. As they would all reduce to the maximal outer approximation $m'(A_i) = \min(\mathscr{L}^i) - \min(\mathscr{L}^{i-1}) = b(A_i) - b(A_{i-1})$ if the power set was totally (rather than partially) ordered, all these consonant approximations can be considered as generalization of the latter. Sufficient conditions on their admissibility can be given in terms of the (partial) plausibility values of the singletons. Due to lack of space, the reader is referred to [24] for the proofs of all main results.

2 Geometry of consonant belief functions

Given a domain Θ, each belief function $b : 2^\Theta \to [0,1]$ is completely specified by its $N-2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$, $N \doteq 2^n$ ($n \doteq |\Theta|$), (as $b(\emptyset) = 0$, $b(\Theta) = 1$ for all b.f.s), and can therefore be represented as a vector $\mathbf{b} = [b(A), \emptyset \subsetneq A \subsetneq \Theta]'$ of $\mathbb{R}^{N-2}$. We can prove that [13, 18] the set of points of $\mathbb{R}^{N-2}$ which correspond to a b.f. or *belief space* $\mathscr{B}$ is the convex closure $\mathscr{B} = Cl(\mathbf{b}_A, \emptyset \subsetneq A \subseteq \Theta)$, where b_A is the *categorical* [7] belief function assigning all the mass to a single subset $A \subseteq \Theta$ and Cl denotes the convex closure operator: $Cl(\mathbf{b}_1, ..., \mathbf{b}_k) = \{\mathbf{b} \in \mathscr{B} : \mathbf{b} =$

$\alpha_1 \mathbf{b}_1 + \cdots + \alpha_k \mathbf{b}_k, \sum_i \alpha_i = 1,\ \alpha_i \geq 0\ \forall i\}$. The belief space $\mathscr{B}$ is a simplex[1] [13], and each vector $\mathbf{b} \in \mathscr{B}$ representing a belief function b can be written as a convex sum as: $\mathbf{b} = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) \mathbf{b}_A$. The set $\mathscr{P}$ of all "Bayesian" b.f.s (assigning non-zero masses to singletons only: $m_b(A) = 0$ if $|A| > 1$) is the simplex[2] $\mathscr{P} = Cl(\mathbf{b}_x, x \in \Theta)$.

In the case of a domain $\Theta_2 = \{x, y\}$ of cardinality 2, each b.f. b is completely determined by its mass values $m_b(x)$, $m_b(y)$, as $m_b(\Theta) = 1 - m_b(x) - m_b(y)$ and $m_b(\emptyset) = 0$, and is represented by a vector $\mathbf{b} = [b(x) = m_b(x), b(y) = m_b(y)]' \in \mathbb{R}^2$.



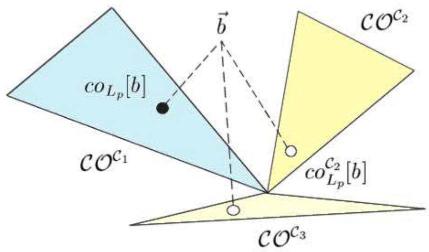


Fig. 1 Left: the belief space $\mathscr{B}_2$ for a binary frame is a triangle in $\mathbb{R}^2$ whose vertices are the vectors $\mathbf{b}_x = [1,0]'$, $\mathbf{b}_y = [0,1]'$, $\mathbf{b}_\Theta = [0,0]'$ associated with the categorical belief functions focused on $\{x\}, \{y\}$ and Θ, respectively. Consonant b.f.s live in the union of the segments $\mathscr{CO}^{\{x,\Theta\}}$ and $\mathscr{CO}^{\{y,\Theta\}}$. The unique $L_1 = L_2$ consonant approximation (circle) and the set of L_∞ consonant approximations (dashed segment) on $\mathscr{CO}^{\{x,\Theta\}}$ are shown. Right: To minimize the distance of a point from a simplicial complex, we need to find all the partial solutions on all the simplices in the complex (empty circles), and compare them to select a global one (black circle).

Since $m_b(x) \geq 0$, $m_b(y) \geq 0$, and $m_b(x) + m_b(y) \leq 1$, the set $\mathscr{B}_2$ of all the possible b.f.s on Θ_2 can be depicted as the triangle in the Cartesian plane of Figure 1-left. The region $\mathscr{P}_2$ of all Bayesian b.f.s on Θ_2 is the diagonal line segment $Cl(\mathbf{b}_x, \mathbf{b}_y)$. On $\Theta_2 = \{x, y\}$ consonant belief functions can have as chain of focal elements either $\{\{x\} \subset \Theta_2\}$ or $\{\{y\} \subset \Theta_2\}$. Therefore, they live in the union of two segments (see Figure 1-left): $\mathscr{CO}_2 = \mathscr{CO}^{\{x,\Theta\}} \cup \mathscr{CO}^{\{y,\Theta\}} = Cl(\mathbf{b}_x, \mathbf{b}_\Theta) \cup Cl(\mathbf{b}_y, \mathbf{b}_\Theta)$.

Approximation in the consonant complex. In the general case the region $\mathscr{CO}$ of consonant belief functions in the belief space is a *simplicial complex* [20, 6], i.e., the union of a collection of (maximal) simplices, each associated with a maximal chain $\mathscr{C} = \{A_1 \subset \cdots \subset A_n\}$, $|A_i| = i$, $A_n = \Theta$ of subsets of Θ:

$$\mathscr{CO} = \bigcup_{\mathscr{C}} \mathscr{CO}^{\mathscr{C}} = \bigcup_{\mathscr{C} = A_1 \subset \cdots \subset A_n} Cl(\mathbf{b}_{A_1}, \cdots, \mathbf{b}_{A_n}).$$

[1] The convex closure $Cl(x_1, ..., x_{n+1})$ of $n+1$ (affinely independent) points $x_1, ..., x_{n+1}$ of $\mathbb{R}^n$ [13].

[2] We denote the categorical b.f. associated with a singleton x by b_x rather than $b_{\{x\}}$, and write $m_b(x), b(x), pl_b(x)$ instead of $m_b(\{x\}), b(\{x\}), pl_b(\{x\})$.

Given a belief function b, we call *consonant approximation of b induced by a distance function d* in $\mathscr{B}$ the b.f.(s) $\mathscr{CO}_d[b]$ which minimize(s) the distance $d(\mathbf{b},\mathscr{CO})$ between b and the consonant simplicial complex in $\mathscr{B}$. We use the notation $co_d[b]$ when the solution is unique, or to denote the barycenter of the set of solutions $\mathscr{CO}_d[b]$. As the consonant complex $\mathscr{CO}$ is a *collection* of simplices which generate distinct linear spaces, solving the approximation problem involves finding first a number of partial solutions: $co^{\mathscr{C}}_{L_p}[b] = \arg\min_{\mathbf{co}\in\mathscr{CO}^{\mathscr{C}}} \|\mathbf{b}-\mathbf{co}\|_{L_p}$ (see Figure 1-right), one for each maximal chain $\mathscr{C}$ of subsets of Θ. Then, the distance of b from all partial solutions has to be assessed in order to select a global optimum.

In recent times, L_p norms have been successfully employed in the probability transformation problem [11] and for conditioning [21, 14]. For vectors $\mathbf{b}$, $\mathbf{b}' \in \mathscr{B}$ representing two belief functions b, b', such norms read as: $\|\mathbf{b}-\mathbf{b}'\|_{L_1} \doteq \sum_{\emptyset\subsetneq B\subseteq\Theta} |b(B)-b'(B)|$; $\|\mathbf{b}-\mathbf{b}'\|_{L_2} \doteq \sqrt{\sum_{\emptyset\subsetneq B\subseteq\Theta}(b(B)-b'(B))^2}$, and $\|\mathbf{b}-\mathbf{b}'\|_{L_\infty} \doteq \max_{\emptyset\subsetneq B\subseteq\Theta} |b(B)-b'(B)|$. Clearly, however, a number of other norms can be picked [22]: this paper is as just a first step of a long line of research.

3 Consonant approximation in the belief space

L_1 approximation. A compact expression for the set of partial L_1 consonant approximations in $\mathscr{B}$ can be found in terms of a list of belief values very much related to the *maximal (partial) outer consonant approximation* [3] with maximal chain $\mathscr{C}$:

$$m_{co^{\mathscr{C}}_{max}[b]}(A_i) = \sum_{B\subseteq A_i, B\not\subseteq A_{i-1}} m_b(B) = b(A_i) - b(A_{i-1}). \qquad (1)$$

Theorem 1. *Given a b.f. $b: 2^\Theta \to [0,1]$, its partial L_1 consonant approximations $\mathscr{CO}^{\mathscr{C}}_{L_1}[b]$ in $\mathscr{B}$ with maximal chain of focal elements $\mathscr{C} = \{A_1 \subset \cdots \subset A_n, |A_i| = i\}$ are the co.b.f.s co whose mass vectors $[m_{co}(A_1),...,m_{co}(A_n)]'$ live in the convex closure:*

$$Cl\Big(\big[b^1, b^2-b^1, \cdots, b^i-b^{i-1}, \cdots, 1-b^{n-1}\big]' \,\Big|\, b^i \in \big\{\gamma^i_{int1}, \gamma^i_{int2}\big\}\ \forall i\Big), \qquad (2)$$

where $\gamma^i_{int1}, \gamma^i_{int2}$ are the innermost (median) elements of the list of belief values:

$$\mathscr{L}_i = \Big\{b(A), A \supseteq A_i, A \not\supset A_{i+1}\Big\}. \qquad (3)$$

As $b^{n-1} = \gamma^{n-1}_{int1} = \gamma^{n-1}_{int2} = b(A_{n-1})$, (2) is a polytope of 2^{n-2} vertices.

Note that, even though we are here seeking approximations in $\mathscr{B}$, we present our results in terms of mass assignments, as they are simpler and easier to interpret. Due to the nature of partially ordered set of 2^Θ, the innermost values of the above lists (3) cannot be analytically identified in full generality (even though they can be easily computed numerically), but can be derived in some simple (e.g. ternary) cases.
As for the *global* L_1 approximation(s):

Theorem 2. *Given a belief function $b : 2^\Theta \rightarrow [0,1]$, its global L_1 consonant approximations $\mathscr{CO}_{L_1}[b]$ in $\mathscr{B}$ live in the collection of partial such approximations associated with the maximal chain(s) $A_1 \subset ... \subset A_n$ which maximize the cumulative lower halves of the lists of belief values $\mathscr{L}_i$ (3):* $\arg\max_{\mathscr{C}} \sum_i \sum_{b(A) \in \mathscr{L}_i, b(A) \leq \gamma^i_{int_1}} b(A)$.

(Partial) L_2 approximation. To find the partial consonant approximation(s) at minimal L_2 distance from b in $\mathscr{B}$ we need to impose the orthogonality of the difference vector $\mathbf{b} - \mathbf{co}$ with respect to any given simplicial component $\mathscr{CO}^{\mathscr{C}}$ of the complex $\mathscr{CO}$: $\langle \mathbf{b} - \mathbf{co}, \mathbf{b}_{A_j} - \mathbf{b}_\Theta \rangle = \langle \mathbf{b} - \mathbf{co}, \mathbf{b}_{A_j} \rangle = 0 \ \ \forall A_j \in \mathscr{C}, 1 \leq j \leq n-1$, as $\mathbf{b}_\Theta = \mathbf{0}$ is the origin of the Cartesian space in $\mathscr{B}$, and $\mathbf{b}_{A_j} - \mathbf{b}_\Theta$ for $j = 1,...,n-1$ are the generators of $\mathscr{CO}^{\mathscr{C}}$ (compare the binary case of Figure 1-left). The L_2 partial approximation of b is unique, and a function of the list of belief values (3) as well.

Theorem 3. *Given a b.f. $b : 2^\Theta \rightarrow [0,1]$, its partial L_2 consonant approximation $co^{\mathscr{C}}_{L_2}[b]$ in $\mathscr{B}$ with maximal chain $\mathscr{C} = \{A_1 \subset \cdots \subset A_n\}$ is unique, and has b.p.a.:*

$$m_{co^{\mathscr{C}}_{L_2}[b]}(A_i) = ave(\mathscr{L}_i) - ave(\mathscr{L}_{i-1}) \qquad \forall i = 1,...,n, \tag{4}$$

where $ave(\mathscr{L}_i) = \frac{1}{2^{|A^c_{i+1}|}} \sum_{A \supseteq A_i, A \not\supset A_{i+1}} b(A)$ *is the average of the list $\mathscr{L}_i$ (3), $\mathscr{L}_0 \doteq \{0\}$.*

L_∞ **approximation** also form a polytope, with 2^{n-1} vertices.

Theorem 4. *Given a belief function $b : 2^\Theta \rightarrow [0,1]$, its partial L_∞ consonant approximations in the belief space $\mathscr{CO}^{\mathscr{C}}_{L_\infty}[b]$ with maximal chain of focal elements $\mathscr{C} = \{A_1 \subset \cdots \subset A_n, |A_i| = i\}$ are the co.b.f.s co whose mass vectors $[m_{co}(A_1), ..., m_{co}(A_n)]'$ live in the convex closure:*

$$Cl\Big(\big[b^1, b^2 - b^1, \cdots, b^i - b^{i-1}, \cdots, 1 - b^{n-1} \big]' \ \Big| \ b^i = \tfrac{b(A_i) + b(\{x_{i+1}\}^c)}{2} + \{ -b(A^c_1), b(A^c_1)\} \ \forall i = 1,...,n-1 \Big). \tag{5}$$

Note that, since $b(A^c_1) = 1 - pl_b(A_1) = 1 - pl_b(x_1)$, the size of the polytope (5) is a function of the plausibility of the innermost desired focal element only. The barycenter $co^{\mathscr{C}}_{L_\infty}[b]$ of (5) has b.p.a.:

$$m_{co^{\mathscr{C}}_{L_\infty}[b]}(A_i) = \frac{b(A_i) - b(A_{i-1})}{2} + \frac{pl_b(x_i) - pl_b(x_{i+1})}{2}, \quad 2 \leq i \leq n-1, \tag{6}$$

while $m_{co^{\mathscr{C}}_{L_\infty}[b]}(A_1) = \frac{b(A_1) + b(\{x_2\}^c)}{2}$, $m_{co^{\mathscr{C}}_{L_\infty}[b]}(A_n) = 1 - b(A_{n-1})$.

Now, let us call *contour-based* consonant approximation of a b.f. b with maximal chain of focal elements $\mathscr{C} = \{A_1 \subset \cdots \subset A_n\}$ the co.b.f. with mass assignment:

$$m_{co_{con}[b]}(A_i) = \begin{cases} 1 - pl_b(x_2) & i = 1, \\ pl_b(x_i) - pl_b(x_{i+1}) & i = 2,...,n-1, \\ pl_b(x_n) & i = n, \end{cases} \tag{7}$$

where $\{x_i\} \doteq A_i \setminus A_{i-1}$ for all $i = 1,...,n$. Such an approximation uses the (unnormalized) contour function of an arbitrary b.f. b to generate a consonant b.f., *as if* it was a possibility distribution. Then, by (1) and (7), it is clear that the barycenter of the partial L_∞ approximations in $\mathscr{B}$ is the *average of the maximal outer consonant approximation and what we called "contour-based" consonant approximation*.

As the distance from b of the partial solutions (5) is $b(A_1^c)$ (see the proof of Theorem 4, [24]), the global L_∞ consonant approximations of b in $\mathscr{B}$ are associated with the chains of focal elements: $\arg\min_{\mathscr{C}} b(A_1^c) = \arg\min_{\mathscr{C}}(1 - pl_b(A_1)) = \arg\max_{\mathscr{C}} pl_b(A_1)$, which are *nested around the maximal plausibility singleton*.

Geometric approximations as generalized maximal outer approximations. From Theorems 1, 3 and 4, the b.p.a.s of *all L_p partial approximations in the belief space are differences of simple functions of belief values taken from the list* (3):

$$
\begin{array}{l}
m_{co^{\mathscr{C}}_{max}[b]}(A_i) = \min(\mathscr{L}_i) - \min(\mathscr{L}_{i-1}); \;\; m_{co^{\mathscr{C}}_{con}[b]}(A_i) = \max(\mathscr{L}_i) - \max(\mathscr{L}_{i-1}); \\
m_{co^{\mathscr{C}}_{L_1}[b]}(A_i) = \frac{int_1(\mathscr{L}_i)+int_2(\mathscr{L}_i)}{2} - \frac{int_1(\mathscr{L}_{i-1})+int_2(\mathscr{L}_{i-1})}{2}; \\
m_{co^{\mathscr{C}}_{L_2}[b]}(A_i) = ave(\mathscr{L}_i) - ave(\mathscr{L}_{i-1}); \\
m_{co^{\mathscr{C}}_{L_\infty}[b]}(A_i) = \frac{\max(\mathscr{L}_i)+\min(\mathscr{L}_i)}{2} - \frac{\max(\mathscr{L}_{i-1})+\min(\mathscr{L}_{i-1})}{2}.
\end{array}
\tag{8}
$$

The maximal outer approximation $co^{\mathscr{C}}_{max}[b]$ is obtained by picking as representative $\min(\mathscr{L}_i)$, $co^{\mathscr{C}}_{con}[b]$ amounts to picking $\max(\mathscr{L}_i)$, the barycenter of the L_1 approximations to choosing the average innermost (median) value, the barycenter of the L_∞ approximations to the average outermost value, L_2 to picking the overall average value of the list. Each vertex of the L_1 solution set (2) amounts to selecting, for each component, either one of the innermost values; each vertex of the L_∞ polytope (5), either one of the outermost values.

Belief functions are defined on a partially ordered set, the power set $2^\Theta = \{A \subseteq \Theta\}$, of which a maximal chain is a maximal totally ordered subset. Therefore, given two elements of the chain $A_i \subset A_{i+1}$, there are a number of "intermediate" focal elements A which contain the latter but not the former. If 2^Θ were to be a totally ordered set, the list $\mathscr{L}_i$ would contain a single element $b(A_i)$ and all the L_p approximations (8) would reduce to the function $co^{\mathscr{C}}_{max}[b]$ (1): they can all be seen as different *generalizations of the maximal outer consonant approximation*.

Graphical comparison in a ternary example. It can be useful to compare the different approximations in the toy case of a ternary frame, $\Theta = \{x,y,z\}$. Let the desired consonant approximation have maximal chain $\mathscr{C} = \{\{x\} \subset \{x,y\} \subset \Theta\}$. Figure 2 illustrates the different partial L_p consonant approximations in $\mathscr{B}$ in the simplex of consonant belief functions with chain $\mathscr{C}$, for a b.f. b with masses: $m_b(x) = 0.2, \;\; m_b(y) = 0.3, \;\; m_b(x,z) = 0.5$. The analogous L_p approximations *in the mass space* [23] (in which b.f.s are represented by their mass vectors) for the same example function are depicted for comparison. Its isopignistic approximation

$$
m_{co_{iso}[b]}(A_i) = i \cdot \big(BetP[b](x_i) - BetP[b](x_{i+1})\big), \; \{x_i\} \doteq A_i \setminus A_{i-1} \forall i \tag{9}
$$

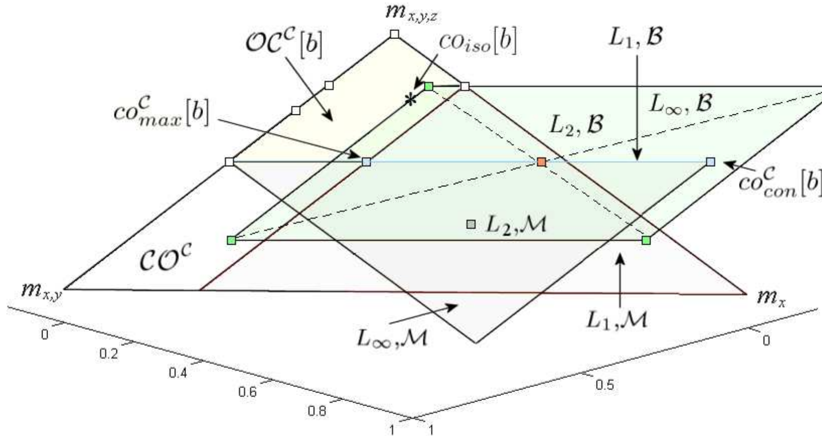


Fig. 2 Comparison between L_p partial consonant approximations in the mass $\mathscr{M}$ and belief $\mathscr{B}$ spaces for the b.f. of the example. The $L_2, \mathscr{B}$ approximation is plotted as a red square, as the barycenter of both the sets of $L_1, \mathscr{B}$ (blue segment) and $L_\infty, \mathscr{B}$ (green quadrangle) approximations. Contour-based and maximal outer approximations are in this example the extreme of the segment $L_1, \mathscr{B}$ (blue squares). The partial outer consonant approximations (yellow), the isopignistic approximation (star) and the various L_p partial approximations in $\mathscr{M}$ (in gray levels) are also drawn.

[8] is also plotted. For the comparison to be homogeneous, we plot both sets of approximations (in the belief and in the mass space) as vectors **m** of mass values. As for the approximations (8) in the belief space, the relevant lists of belief values are $\mathscr{L}_1 = \{b(x), b(x,z)\}$ and $\mathscr{L}_2 = \{b(x,y)\}$, so that $\min(\mathscr{L}_1) = int_1(\mathscr{L}_1) = b(x)$, $\max(\mathscr{L}_1) = int_2(\mathscr{L}_1) = b(x,z)$, $ave(\mathscr{L}_1) = \frac{b(x)+b(x,z)}{2}$; $\min(\mathscr{L}_2) = int_1(\mathscr{L}_2) = \max(\mathscr{L}_2) = int_2(\mathscr{L}_2) = ave(\mathscr{L}_2) = b(x,y)$. Therefore, the set of L_1 partial consonant approximations is, by Equation (2), a segment with vertices:

$$\big[b(x), b(x,y) - b(x), 1 - b(x,y)\big]', \ \big[b(x,z), b(x,y) - b(x,z), 1 - b(x,y)\big]' \qquad (10)$$

(the blue segment in Figure 2). The partial L_2 approximation in $\mathscr{B}$ is, by Equation (8), unique (red square) and coincides with the barycenter of the set of partial L_∞ approximations (green quadrangle). This is not so in the general case.

Admissibility. The example shows that geometric approximation in the belief space generates solutions which are in general only partially admissible, i.e., they may contain approximations with negative masses. While computing their admissible part is not trivial, sufficient conditions on the desired maximal chain under which they are indeed admissible can be given in terms of the list of belief values (3). As $\min(\mathscr{L}_{i-1}) = b(A_{i-1}) \leq b(A_i) = \min(\mathscr{L}_i)$, the maximal partial outer approximation co_{max} is admissible for all maximal chains $\mathscr{C}$. As for the contour-based approximation co_{con}, $\max(\mathscr{L}_i) = b(A_i + A^c_{i+1}) = b(x^c_{i+1}) = 1 - pl_b(x_{i+1})$ (when once again $x_i \doteq A_i \setminus A_{i-1}$), while $\max(\mathscr{L}_{i-1}) = 1 - pl_b(x_i)$, so that $\max(\mathscr{L}_i) - \max(\mathscr{L}_{i-1}) = pl_b(x_i) - pl_b(x_{i+1})$, which is guaranteed non-negative *if the chain $\mathscr{C}$ is generated by singletons sorted by their plausibility values*. As a consequence, the barycenter of the set of L_∞ approximations is also admissible on the same chain(s). A similar sufficient condition holds in the L_1, L_2 cases [24].

4 Conclusions

Approximations in the mass and the belief space turn out to be inherently related to completely different philosophies to the consonant approximation problem: mass redistribution versus generalized maximal outer approximation. Isopignistic, mass-space and belief-space consonant approximations form three distinct families of approximations, with fundamentally different rationales: which approach to use will therefore vary according to the chosen framework, and the problem at hand.

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