# Generalizations of the relative belief transform

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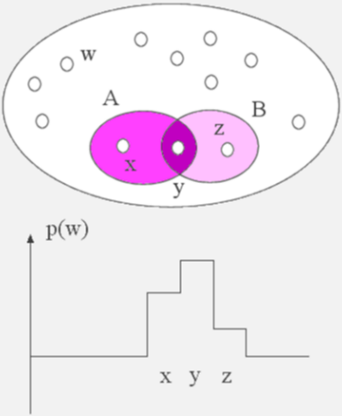
# Outline of the talk




1. Probability trasformation
2. The relative belief transform
3. Generalizing the relative belief transform
4. Discussion in two scenarios
5. Conclusions

# Belief versus probability measures

- a **belief function** $b : 2^\Theta \rightarrow [0,1]$ is such that,

$$b(A) = \sum_{B \subseteq A} m(B)$$

where $m : 2^\Theta \rightarrow [0,1]$ is a basic belief (probability) assignment s.t.

$$m(\emptyset) = 0, \sum_{A \subseteq \Theta} m(A) = 1, m(A) \geq 0 \ \forall A \subseteq \Theta$$

- **Bayesian** belief functions: all focal elements are singletons
- are in 1-1 correspondence with (finite) probability measures

# Mapping a belief function to a possibility

- the relation between belief and probability, in particular, has been an important subject of study in the theory of evidence
- **probability transform** of belief functions: an operator $pt : \mathcal{B} \rightarrow \mathcal{P}$, $b \mapsto pt[b]$ mapping belief measures onto probability distributions
- not necessarily “consistent” with original b.f., i.e., an element of the corresponding credal set
- they are **less computationally expensive** than b.f.s
- allow to **make decisions in the expected utility framework**
- a number of transforms proposed, either as efficient implementations of ToE or tools for decision making

# Some literature on probability transforms

- very much not exhaustive!
- [*Lowrance* 86] “summarization” technique
- [*Smets* 88] pignistic transform
- [*Voorbraak* 89] plausibility transform
- [*Tessem* 93] $m_{klx}$ approximation incorporating only the highest-valued focal elements
- [*Cobb&Shenoy* 03] plausibility transform and Dempster’s rule
- [*Sudano* 03], [*Dezert* 07], [*Burger* 10] redistribution processes similar to that of the pignistic transform
- [*Daniel* 06] normalized belief of singletons
- [*Cuzzolin* 07] orthogonal projection and intersection probability
- [*Cuzzolin* 10] geometry of plausibility and belief transform

# Transformation in the TBM: the pignistic transform

- in Smets' Transferable Belief Model a probability transform is central
- **pignistic probability**

$$BetP[b](x) = \sum_{A \supseteq \{x\}} \frac{m_b(A)}{|A|},$$

  the result of a *pignistic transform*
- its purpose is to allow decision making at the level of probabilities, typically in an expected utility framework
- the result of a redistribution process in which the mass of each focal element $A$ is re-assigned to all its elements $x \in A$ on an equal basis
- it commutes with affine combination and is the center of mass of the credal set of consistent probabilities

# Plausibility transform

- other transforms have been proposed for efficient implementation of belief calculus
- plausibility transform [Voorbraak 89]
- maps a belief function to the **relative belief of singletons**:

$$\tilde{pl}_b(x) = \frac{pl_b(x)}{\sum_{y \in \Theta} pl_b(y)}$$

- relative plausibility of singletons $\tilde{pl}_b$ is a perfect representative of $b$ when combined with other probabilities $p \in \mathcal{P}$ by Dempster's rule $\oplus$
- meets a number of properties w.r.t. $\oplus$ [Coob&Shenoy 03]

# Relative belief transform

- the *relative belief transform* $\tilde{b} : \mathcal{B} \rightarrow \mathcal{P}$, $b \mapsto \tilde{b}[b]$ ...
- ... maps each belief function to the corresponding *relative belief of singletons*:

$$\tilde{b}(x) = \frac{b(x)}{\sum_{y \in \Theta} b(y)}$$

- it exists iff $b$ assigns some mass to singleton focal sets: $\sum_{x \in \Theta} m_b(x) \neq 0$
- first been proposed by Daniel in 2006
- its geometry and that of plausibility transform analyzed in [Cuzzolin 2010 AMAI]

# Properties and duality

- a similar set of (dual) properties hold for the relative belief transform [Cuzzolin 12 IJAR]

$$pl_b \oplus p = \tilde{b} \oplus p \; \forall p; \quad \tilde{b}[pl_{b_1} \oplus pl_{b_2}] = \tilde{b}[pl_{b_1}] \oplus \tilde{b}[pl_{b_2}];$$
$$pl_b \oplus pl_b = pl_b \vdash \tilde{b}[pl_b] \oplus \tilde{b}[pl_b] = \tilde{b}[pl_b],$$

- $pl_b \oplus$ denotes the extension of Dempster's rule to plausibility measures, seen as pseudo belief functions
- there exists a family of probability transformations strongly linked to Shafer's interpretation of the theory of evidence via Dempster's rule: the **epistemic family** of transformation
- in opposition to the **affine family** of mappings which commute with affine combination [Smets, Cuzzolin 07 SMCB]

# Issues with the relative belief transform

- there is a duality between relative plausibility and belief transform
- however, a serious issue with the relative belief of singletons is its applicability
- $\tilde{b}$ does not exist for a large class of belief functions
- in consonant belief functions at most one focal element is a singleton, therefore the vast majority of the useful information in the b.b.a. is contained in the non-singleton focal elements

# A family of "relative mass" transforms

- relative belief is in fact only one element of an entire family of probability transformations
- can be thought of as the transform which, given a belief function $b$:

  1. retains the focal elements of size 1 only, yielding an unnormalized belief function
  2. computes (indifferently) the latter's relative plausibility/pignistic transformation:

$$\tilde{b}(x) = \frac{\sum_{A \supseteq x, |A|=1} m_b(A)}{\sum_y \sum_{A \supseteq x, |A|=1} m_b(A)} = \frac{m_b(x)}{k_{m_b}} = \frac{\sum_{A \supseteq x, |A|=1} \frac{m_b(A)}{|A|}}{\sum_y \sum_{A \supseteq x, |A|=1} \frac{m_b(A)}{|A|}}$$

# A family of "relative mass" transforms

- natural generalizations of relative belief are obtained by
  1. retaining the focal elements of size $s$ only;
  2. computing either the resulting relative plausibility ...
  3. ... or the associated pignistic transformation.
- indeed, both alternatives *yield the same probability distribution*

## Definition

We call *relative mass transformation* of level $s$ the transform $\tilde{M}_s[b]$ which maps a b.f. $b$ to the probability

$$p(x) = \frac{\sum_{A \supseteq \{x\}: |A| = s} \frac{m_b(A)}{|A|}}{\sum_{y \in \Theta} \sum_{A \supseteq \{y\}: |A| = s} \frac{m_b(A)}{|A|}}$$

and denote by $\tilde{m}_s$ the output of the relative mass transform of level $s$.

# Decomposition of pignistic and plausibility transforms

## into relative mass transforms

- let $k_{b,s} = \sum_{A\subseteq\Theta:|A|=s} m_b(A)$, $pl_b(x;s) = \sum_{A\supseteq\{x\}:|A|=s} m_b(A)$
- for the relative plausibility of singletons we get the following convex decomposition into relative mass probabilities $\tilde{m}_s$:

$$\tilde{pl}_b(x) \;=\; \sum_s \alpha_s \tilde{m}_s(x),$$

  where $\alpha_s = \frac{sk_{b,s}}{\sum_r rk_{b,r}} \propto sk_{b,s} = \sum_y pl_b(y;s)$ measures for each level $s$ the total plausibility contribution of the focal elements of size $s$
- in the case of the pignistic probability we get:

$$BetP[b](x) \;=\; \sum_s k_{b,s}\tilde{m}_s(x)$$

  with $\beta_s = k_{b,s}$ measuring for each level $s$ the mass contribution of the focal elements of size $s$

# Two approximation criteria

- two natural criteria for approximating $\tilde{pl}$, *BetP* via the relative mass transforms:
- (C1) in the convex decompositions associated with $\tilde{pl}$ and *BetP* we retain the component *s* **whose coefficient** ($\alpha_s$ in the first case, $\beta_s$ in the second) **is the largest**
- note that the optimal (C1) approximations of plausibility or pignistic transform are in principle distinct
- (C2) we retain the component associated with the **minimal size focal elements**

# Two approximation criteria

- which is the right one?
- they favor different aspects of the original belief function
- (C1) focuses on the **strength of the evidence** carried by focal elements of equal size
- (C2) favors instead the **precision** of such pieces of evidence

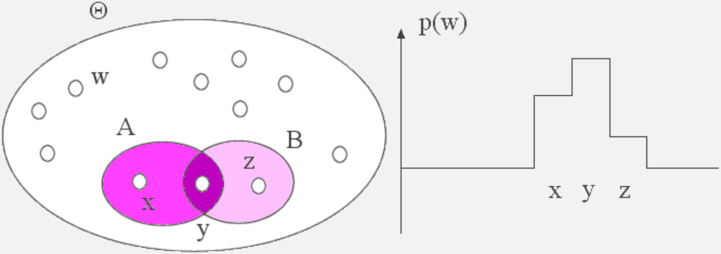
# Natural extension of the relative belief transform

- the relative belief transformation coincides with the approximation produced by (C2) if $\sum_x m_b(x) \neq 0$
- therefore, **(C2) generalizes the relative belief transform**
- when the mass of singletons is nil, instead, the second criterion delivers a **natural extension** of the relative belief operator:

$$\tilde{b}^{ext}(x) \doteq \frac{\sum_{A \supseteq \{x\}:|A|=min} m_b(A)}{|A|_{min} \sum_{A \subseteq \Theta:|A|=min} m_b(A)}.$$

# Scenario 1

- consider a scenario in which we want to approximate the plausibility/pignistic transform of a belief function with b.b.a. s.t. $m_b(A)$ and $m_b(B)$ are small, $|A| = |B| = 2$, while $m_b(\Theta) \gg m_b(A)$




- relative plausibility of singletons:

$$\tilde{pl}_b(x) \propto m_b(A) + m_b(\Theta), \quad \tilde{pl}_b(y) \propto m_b(A) + m_b(B) + m_b(\Theta),$$
$$\tilde{pl}_b(z) \propto m_b(B) + m_b(\Theta), \quad \tilde{pl}_b(w) \propto m_b(\Theta) \ \forall w \neq x, y, z.$$

# Scenario 1 - continued

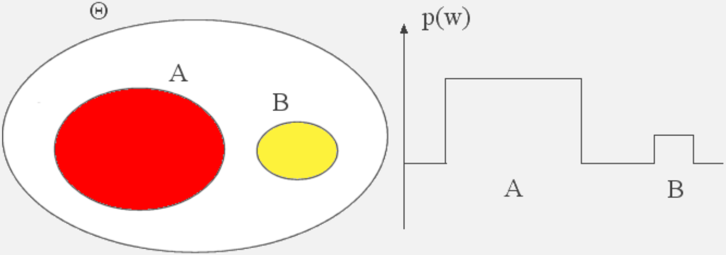
- pignistic probability:

$$BetP(x) = \frac{m_b(A)}{2} + \frac{m_b(\Theta)}{n}, \quad BetP(y) = \frac{m_b(A)+m_b(B)}{2} + \frac{m_b(\Theta)}{n},$$
$$BetP(z) = \frac{m_b(B)}{2} + \frac{m_b(\Theta)}{n}, \quad BetP(w) = \frac{m_b(\Theta)}{n} \forall w \neq x, y, z.$$

- assuming $m_b(A) > m_b(B)$, both transformations have a profile as above
- criterion (C1) yields the average probability $\tilde{m}_1(w) = 1/n \, \forall w \in \Theta$, **no information at all!**
- criterion (C2) yields a probability with the **same profile as the original transforms!**

# Scenario 2

- consider a belief function like this: $m_b(A) \gg m_b(B)$, $|A| > |B|$




- relative plausibility and pignistic probability

$$\tilde{pl}_b(w) = BetP(w) \propto m_b(A) \; w \in A,$$
$$\tilde{pl}_b(w) = BetP(w) \propto m_b(B) \; w \in B,$$

- (C1) generates the uniform probability on elements of $A$
- (C2) generates the uniform probability on elements of $B$
- **(C1) yields the best decision-making approximation** of both

# Conclusions

- we tried and enrich our understanding of the family of epistemic transforms of belief functions
- relative belief is only a member of a class of **relative mass transformations**
- these:
    - **generalize it**
    - are **computable even when the mass of singletons is nil**
    - can be interpreted as **low-cost proxies** for both plausibility and pignistic transforms
- the best approximation criteria varies with the scenario