On the relationship between the notions of independence in matroids, lattices, and Boolean algebras

Fabio Cuzzolin
INRIA Rhône-Alpes
Fabio.Cuzzolin@inrialpes.fr

We discuss the relationship between the notion of independence as defined in lattice theory, matroid theory, and Boolean algebras. Collections of Boolean sub-algebras $\{A_t\}$ are indeed endowed with the following independence relation ($\mathcal{IB}$)

$$\cap A_t \neq \wedge$$

with $\wedge$ the initial element of the Boolean algebra they belong to.
However, those collections can be given several algebraic interpretations in terms of semimodular lattices, matroids, and geometric lattices. Each of those structures are endowed with a particular notion of independence.

On one side they possess the algebraic structure of semimodular lattice, on whose atoms matroidal independence can be introduced and extended to all elements of the lattice in different guises. Even though neither of them coincides with $\mathcal{IB}$, they do possess interesting meaningful interactions. It can be in fact proven that $\mathcal{IB}$ is *not* a form of matroidal independence at all.

On the other side, collections of Boolean sub-algebras form also geometric lattices. Even though they are not matroids in themselves, then, they can be seen as flats of some matroid. A definition of "independence of flats" can be proposed and the possibility that it corresponds to $\mathcal{IB}$ discussed.
Unfortunately, the matroid associated with such a collection is trivial, i.e. all Boolean sub-algebras are independent in this sense.

Indeed, the case of Boolean sub-algebras of cardinality 2 tells us that binary frames are independent as Boolean sub-algebras iff they are *not* independent as elements of the corresponding matroid, and their presence forbids the semimodularity of the associated lattice. In rough words, $\mathcal{IB}$ collections form "anti-matroids".

MSC2000: 03G05, 05B35, 06C10.

Keywords: independence, matroids, semimodular lattices, Boolean algebras.

.