18^{th} British Combinatorial Conference

LATTICE MODULARITY AND LINEAR INDEPENDENCE

Fabio Cuzzolin

DEPARTMENT OF ELECTRONICS AND COMPUTER SCIENCE

UNIVERSITY OF PADOVA

July 1-6, 2001

OUTLINE OF THE TALK

- ***families of frames*** **in the theory of evidence**
- ***lattice structure*** **of the families of frames**
- ***a comparison*****: subspace and frame lattices**
- **independence on *Birkhoff lattices***
  - abstract linear dependence
  - atom independence for semimodular lattices
- ***candidate independence relations*** **for generic elements**
  - on frames and subspaces
- ***modularity*** **and candidate independence relations**
- **an *equivalent condition* for modularity**

FAMILIES OF COMPATIBLE FRAMES

- given two **frames** (finite sets) Θ and Ω, a map $\omega : 2^\Theta \rightarrow 2^\Omega$ is a *refining* if:

  1. $\omega(\{\theta\}) \neq \emptyset \qquad \forall \theta \in \Theta$;
  2. $\omega(\{\theta\}) \cap \omega(\{\theta'\}) = \emptyset \qquad \text{if } \theta \neq \theta'$;
  3. $\cup_{\theta \in \Theta} \omega(\{\theta\}) = \Omega$

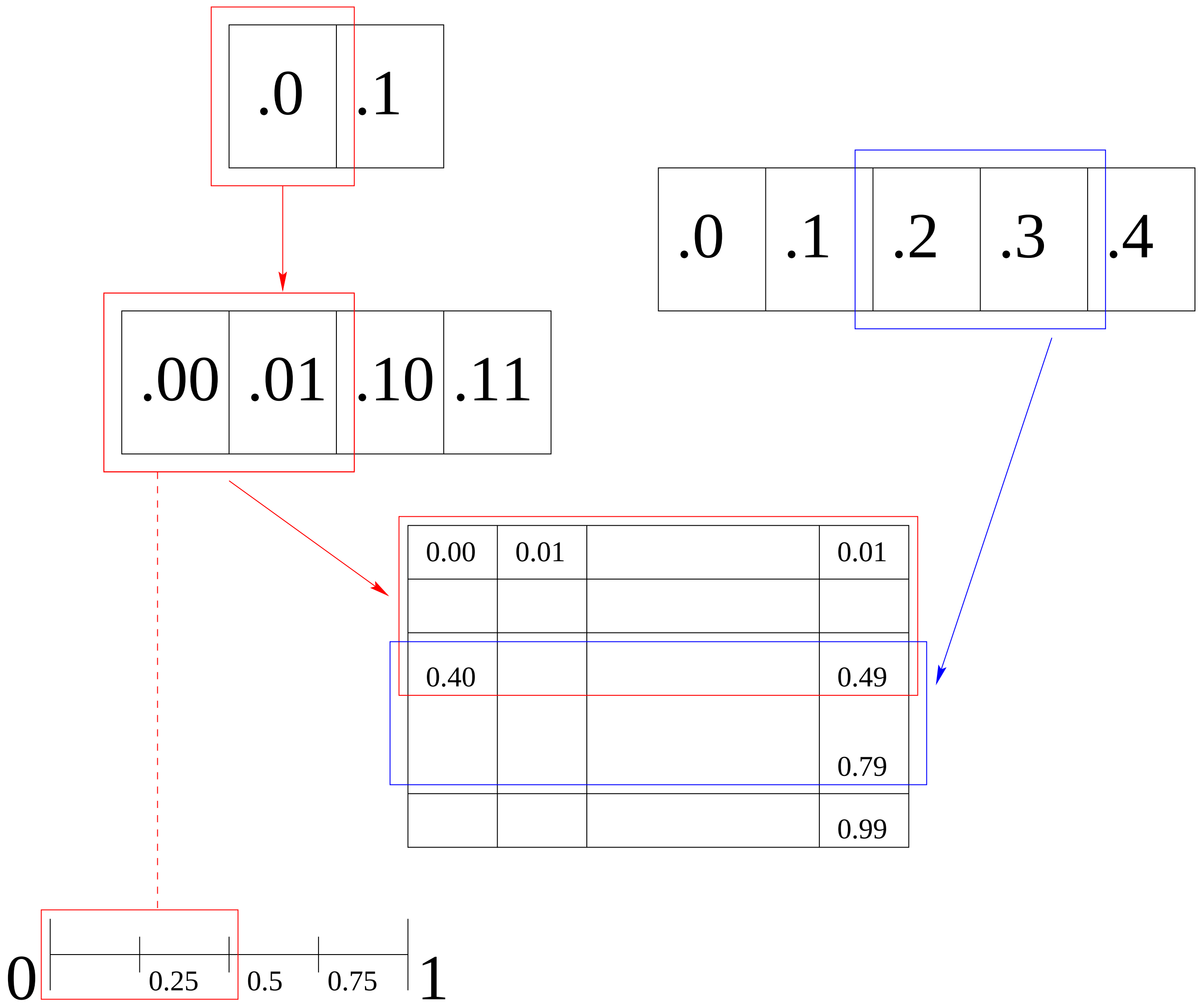
- if $\Theta_1, ..., \Theta_n$ are elements of a family $\mathcal{F}$ then there exists a unique element (**minimal refinement**) $\Theta \in \mathcal{F}$ s.t.

  1. $\forall i \; \exists \; \omega_i : 2^{\Theta_i} \rightarrow 2^\Omega$ refining;
  2. $\forall \theta \in \Theta \; \exists \; \theta_i \in \Theta_i \; for \; i = 1, ..., n \; such \; that$
  $$\{\theta\} = \omega_1(\{\theta_1\}) \cap ... \cap \omega_n(\{\theta_n\})$$

- **family of frames**: non-empty collection of finite non-empty sets $\mathcal{F}$ satisfying

  1. *composition of refinings*
  2. *identity of coarsenings*
  3. *identity of refinings*
  4. *existence of coarsenings*
  5. *existence of refinings*
  6. *existence of common refinements.*

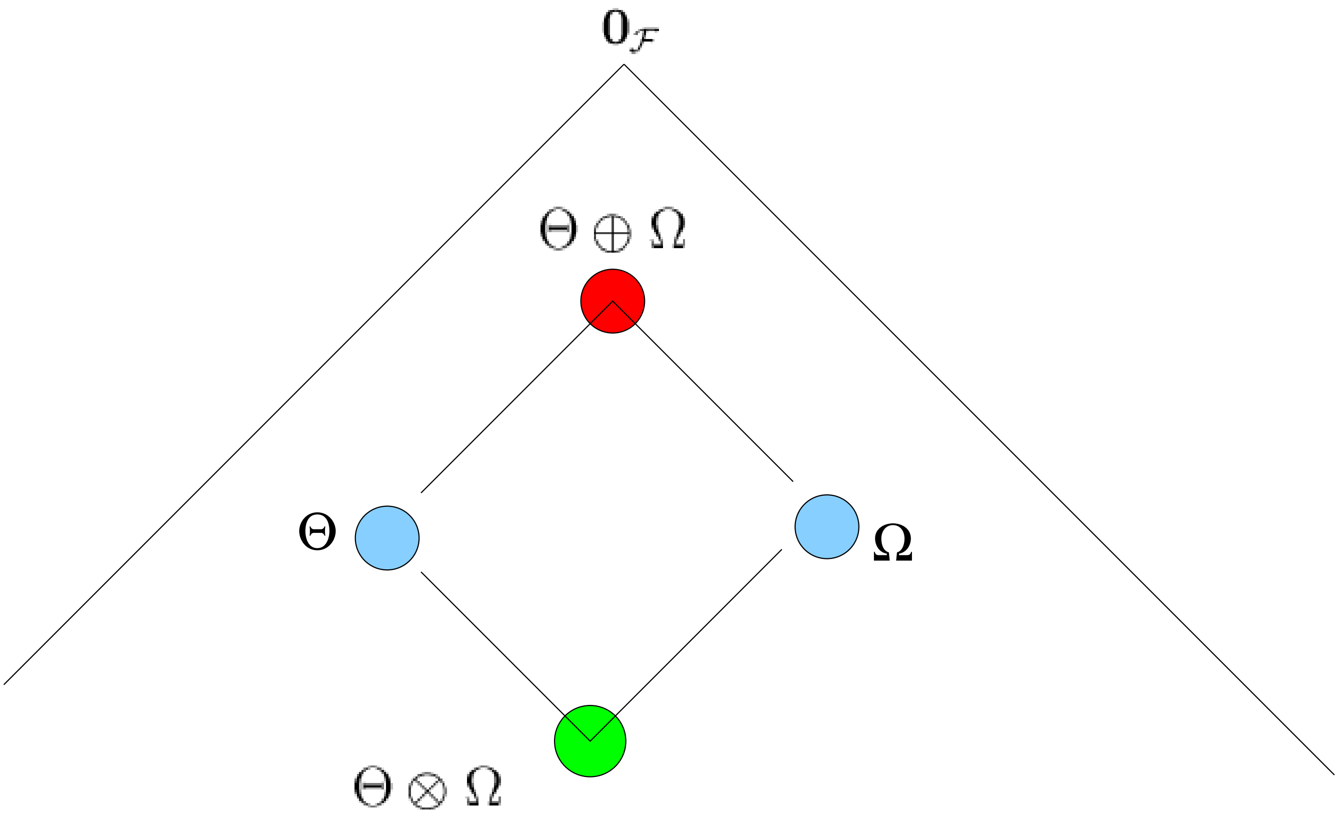
FAMILIES OF COMPATIBLE FRAMES



- **example of compatible frames**: a number between 0 and 1 is expressed in several different basis, with different precisions

- **minimal refinement** of a collection $\Theta_1, ..., \Theta_n$: roughest frame which is refinement of all the frames

LATTICE STRUCTURE OF THE FAMILIES OF FRAMES



- **order relation**

$$\Theta_2 \geq \Theta_1 \equiv \exists\, \Theta_3 \; s.t. \; \Theta_2 = \Theta_1 \otimes \Theta_3 \equiv \exists\, \omega : 2^{\Theta} \rightarrow 2^{\Omega}$$

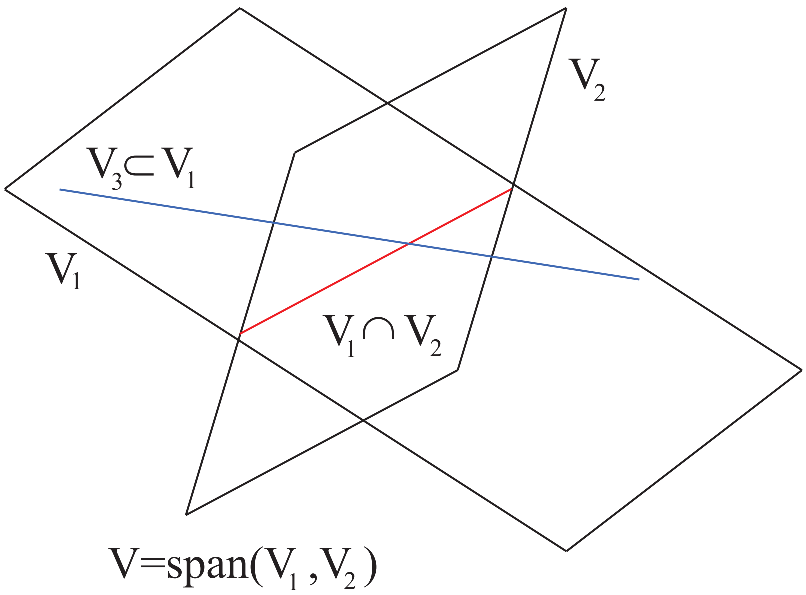
- sup

$$\sup_{\mathcal{F}}(\{\Theta_1, ..., \Theta_n\}) = \Theta_1 \otimes \cdots \otimes \Theta_n$$

- a family of compatible frames $\mathcal{F}$ is a **locally Birkhoff lattice bounded below**, i.e. is a semimodular lattice of locally finite length with initial element

COMPARING SUBSPACE AND FRAME LATTICES

- the collection of subspaces of a linear space is a semimodular lattice



	$L(V)$	$L(\Theta)$
initial element $\mathbf{0}$	$\{\emptyset\}$	$\mathbf{0}_{\mathcal{F}}$
least upper bound $a \vee b$	$span(V, W)$	$\Theta \otimes \Omega$
greatest lower bound $a \wedge b$	$V \cap W$	$\Theta \oplus \Omega$
order relation $a \geq b$	$V \supset W$	$\exists\, \omega : 2^{\Theta} \rightarrow 2^{\Omega}$
rank $h(a)$	$dim(V)$	$\|\Theta\|$

INDEPENDENCE OF FRAMES AND SUBSPACES

- $\Theta_1, ..., \Theta_n$ compatible frames, and
$$\omega_i : 2^{\Theta_i} \to 2^{\Theta_1 \otimes \cdots \otimes \Theta_n}$$
the corresponding refinings to their minimal refinement

- $\Theta_1, ..., \Theta_n$ are called **independent** if
$$\omega_1(A_1) \cap \cdots \cap \omega_n(A_n) \neq \emptyset$$
whenever $\emptyset \neq A_i \subset \Theta_i$ for $i = 1, ..., n$

- independence conditions on vectors and frames show a **similarity**

$$\begin{array}{lcc} \sum_i v_i \neq 0 & \Leftrightarrow & v_1 + ... + v_n \neq 0, \ \forall v_i \in V_i \\ & & \Updownarrow \\ \bigcap_i V_i = \emptyset & \Leftrightarrow & span\{V_1, ..., V_n\} = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n} \end{array}$$

$$\begin{array}{lcc} \bigoplus_i \Theta_i = \mathbf{0}_{\mathcal{F}} & \Leftrightarrow & \Theta_1 \otimes \cdots \otimes \Theta_n = \Theta_1 \times \cdots \times \Theta_n \\ & & \Updownarrow \\ \bigcap A_t \neq \bigwedge & \Leftrightarrow & \omega_1(A_1) \cap \cdots \cap \omega_n(A_n) \neq \emptyset \end{array}$$

ABSTRACT LINEAR DEPENDENCE ON ATOMS OF BIRKHOFF LATTICES

- if L is a lattice bounded below then its **atoms** are the elements of L covering 0, namely

$$A = \{a \in L | a \succ 0\}$$

- consider the set $\mathcal{F}(M)$ of all the finite subsets of a given set M and define in the product $M \times \mathcal{F}(M)$ a relation Λ

- Λ is said to be a **linear dependence** on the set M when

1. $p_j \; \Lambda \; \{p_1, ..., p_m\}, \quad j = 1, ..., n$

2. if $p \; \Lambda \; \{p_1, ..., p_m\}$ and $\forall \; j, \; p_j \; \Lambda \; \{q_1, ..., q_n\}$

   then $p \; \Lambda \; \{q_1, ..., q_n\}$

3. if $p \; \Lambda \; \{p_1, ..., p_m, q\}$ but $p \; \Lambda' \; \{p_1, ..., p_m\}$ then

   $q \; \Lambda \; \{p_1, ..., p_m, p\}$

CANDIDATE INDEPENDENCE RELATIONS FOR GENERIC ELEMENTS

- 1. $\{p_1, ..., p_n\}$ are $\mathcal{LI}_1$**-independent** if

$$p_j \not\leq \bigvee_{i \neq j} p_i \ \forall j \quad (\equiv p_j \wedge \bigvee_{i \neq j} p_i \neq p_j \ \forall j)$$

  2. $\{p_1, ..., p_n\}$ are $\mathcal{LI}_2$**-independent** if

$$p_k \wedge (p_1 \vee \cdots \vee p_{k-1}) = 0, \ k = 2, ..., n$$

  3. $\{p_1, ..., p_n\}$ are $\mathcal{LI}_3$**-independent** if

$$h(p_1 \vee \cdots \vee p_n) = h(p_1) + ... + h(p_n)$$

- the restrictions of the above relations to the set of the atoms A of a lattice **are equivalent**, namely

$$\{a_1, ..., a_n\} \ \mathcal{LI}_1 \equiv \{a_1, ..., a_n\} \ \mathcal{LI}_2 \equiv \{a_1, ..., a_n\} \ \mathcal{LI}_3$$

  and **their complement is a linear dependence relation**

CANDIDATE RELATIONS ON FRAMES AND SUBSPACES

- on *vector* subspaces (**modular** lattice)

1. $\mathcal{LI}_1$: $V_j \not\subset span(V_1, ..., \hat{V}_j, ..., V_n)\ \forall j$

2. $\mathcal{LI}_2$: $V_k \cap span(V_1, ..., V_{k-1}) = \emptyset$ for $k = 2, ..., n$

3. $\mathcal{LI}_3$: $dim(span(V_1, ..., V_n)) = dim(V_1) + ... + dim(V_n)$

- on *frames* (**semimodular** lattice)

1. $\mathcal{LI}_1$: Θ_j is not refinement of $\Theta_1 \otimes ...\hat{\Theta}_j ... \otimes \Theta_n$

2. $\mathcal{LI}_2$: $\Theta_k \oplus (\Theta_1 \otimes ... \otimes \Theta_{k-1}) = \mathbf{0}_{\mathcal{F}}$ for $k = 2, ..., n$

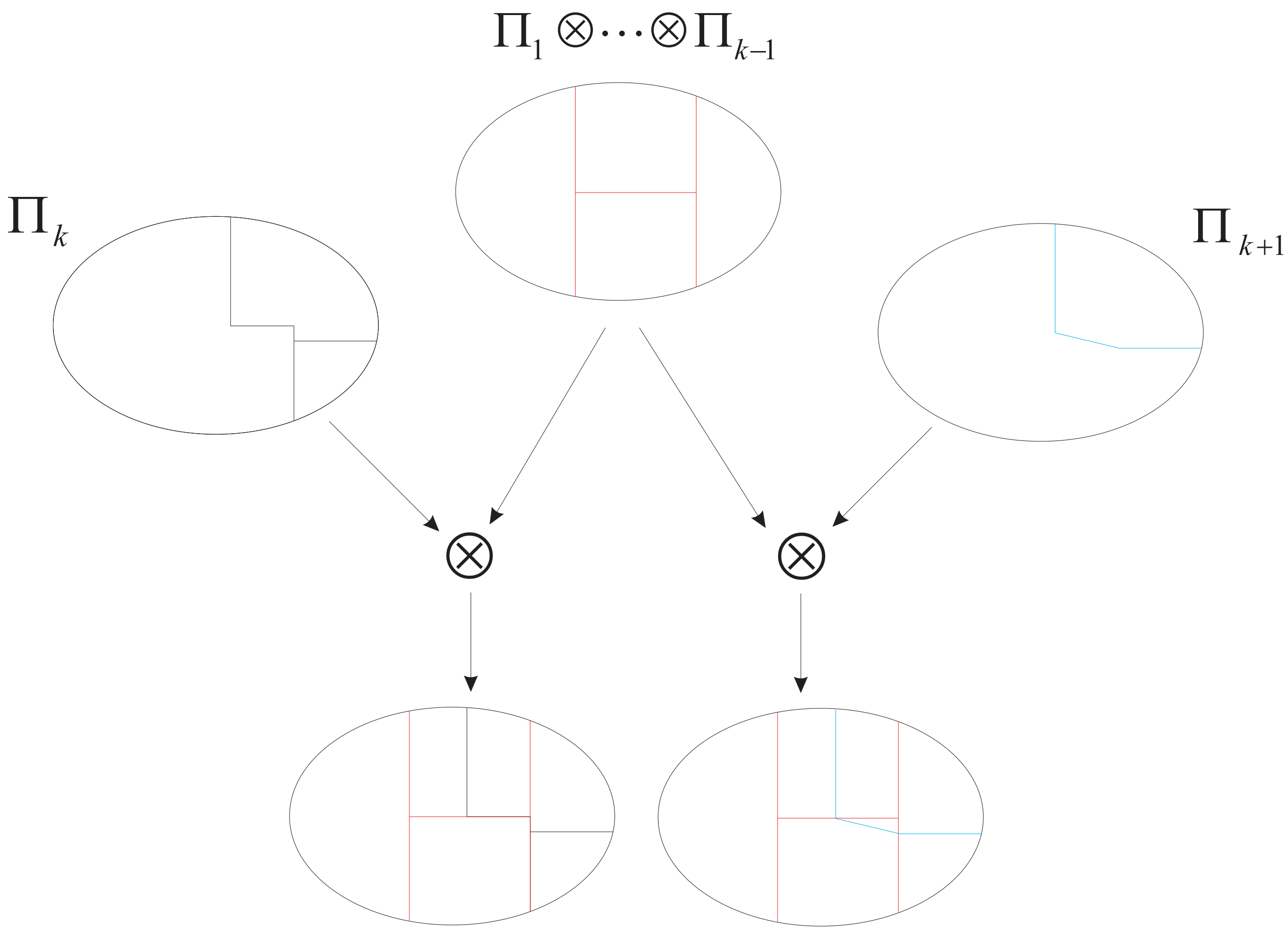
3. $\mathcal{LI}_3$: $|\Theta_1 \otimes ... \otimes \Theta_n| = \sum_i |\Theta_i|$

- for $L(V)$: $\mathcal{LI}_2 \equiv \mathcal{LI}_3$, $\mathcal{LI}_2 \Rightarrow \mathcal{LI}_1$ but $\mathcal{LI}_1 \not\Rightarrow \mathcal{LI}_2$

- for $L(\Theta)$, instead, $\mathcal{LI}_2 \not\Rightarrow \mathcal{LI}_1$

$\mathcal{LI}_2 \not\Rightarrow \mathcal{LI}_1$: A COUNTEREXAMPLE

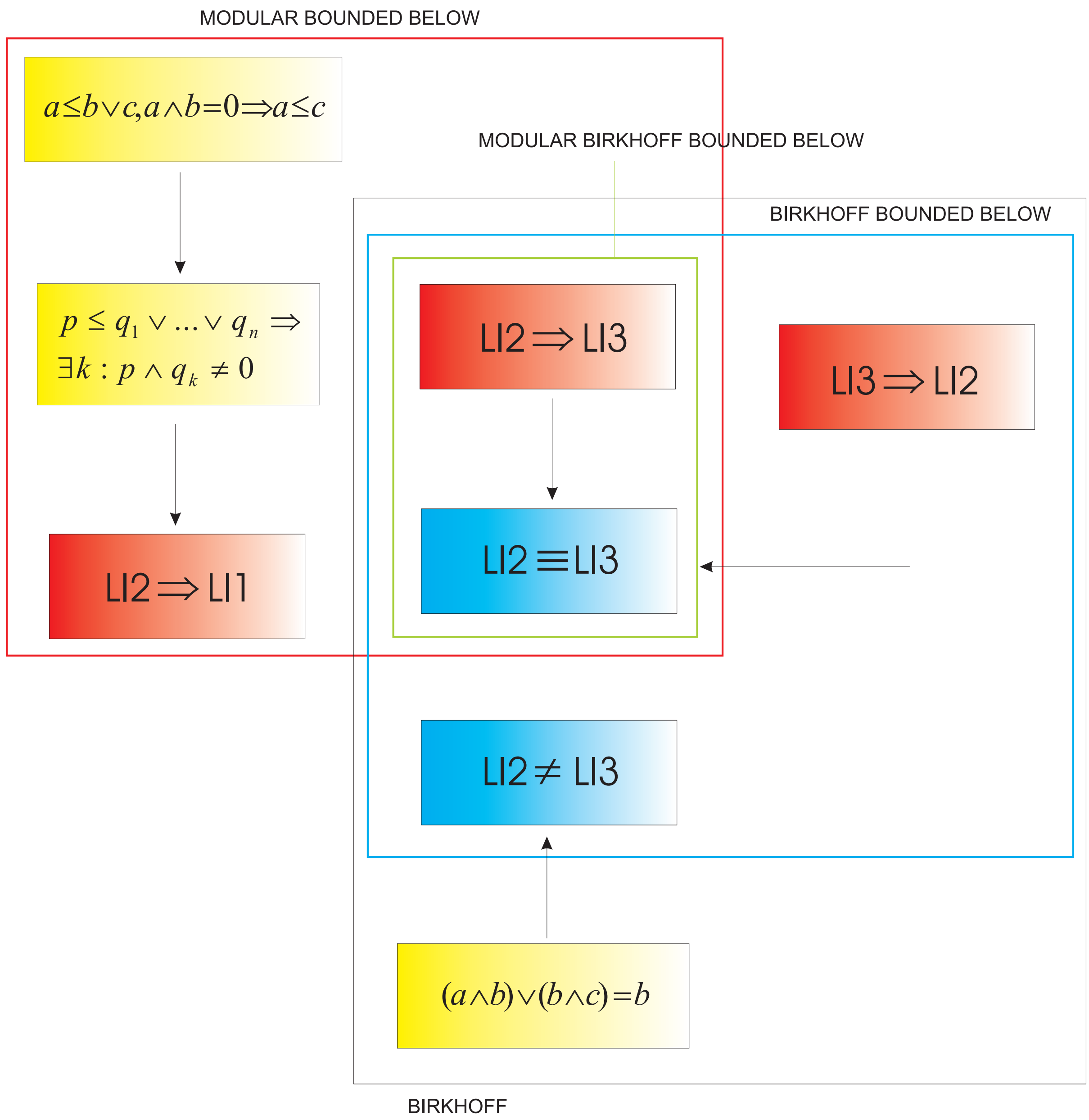
- a counterexample for the conjecture $\mathcal{LI}_2 \Rightarrow \mathcal{LI}_1$ in the frame lattice



- the above diagrams represent, given $P_1 \otimes \cdots \otimes P_{k-1}$ (top set) and P_k, one choice of P_{k+1} contradicting the hypothesis

SCHEME OF THE PROOF

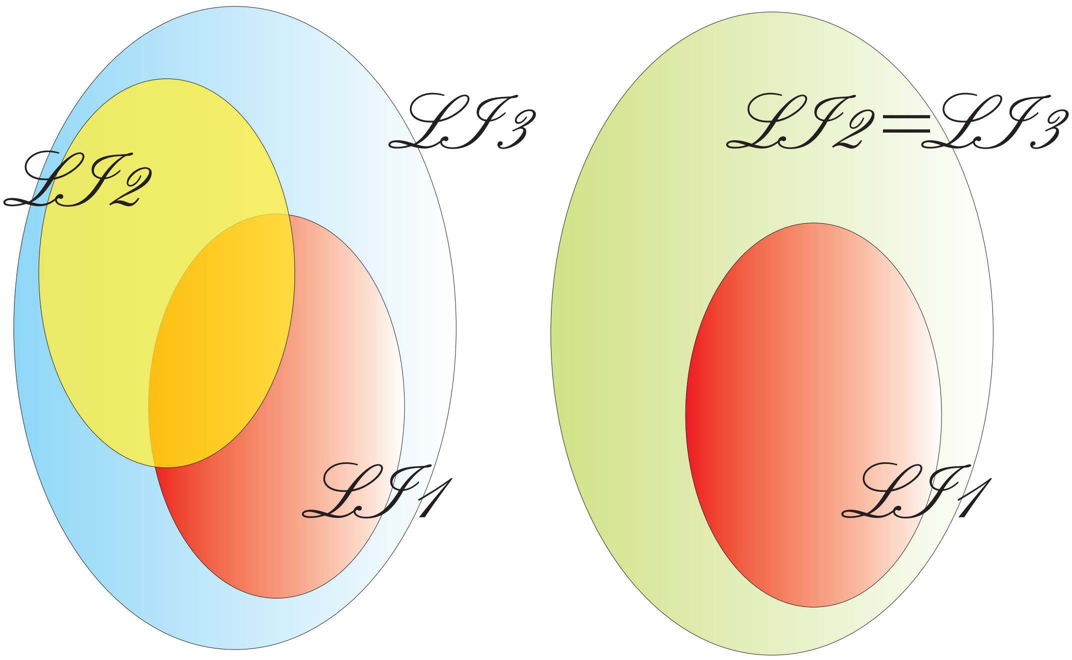
- this suggests a connection between the candidate relations and the property of the underlying lattice



MODULARITY AND CANDIDATE RELATIONS

AN EQUIVALENT CONDITION FOR MODULARITY

- relations among $\mathcal{LI}_1$, $\mathcal{LI}_2$ and $\mathcal{LI}_3$ for semimodular (left) and modular (right) lattices



CONCLUSIONS AND PERSPECTIVES

- the algebraic structure of the collections of Boolean algebras suggests **a connection between linear dependence and modularity**