Lattice modularity and linear independence

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The notion of abstract linear independence on lattices (due to Birkhoff) generalizes the important concept of independence of vectors of a linear space. This relation can be expressed in three different equivalent ways for atoms of a semimodular lattice bounded below, but are in general distinct when applied to arbitrary elements (when they can fail to be independence relations). Let $L$ be a semimodular lattice with initial element, and define $\mathcal{LI}_1, \mathcal{LI}_2, \mathcal{LI}_3$ in the following way:

1. \( \{p_1,...,p_n\} \subset L \) are $\mathcal{LI}_1$-independent if \( p_j \not\leq \bigvee_{i \neq j} p_i \ \forall j \) (\( \equiv p_j \land \bigvee_{i \neq j} p_i \neq p_j \ \forall j \));

2. \( \{p_1,...,p_n\} \subset L \) are $\mathcal{LI}_2$-independent if \( p_k \land (p_1 \lor \cdots \lor p_{k-1}) = 0 \), \( k = 2,...,n \);

3. \( \{p_1,...,p_n\} \subset L \) are $\mathcal{LI}_3$-independent if \( h(p_1 \lor \cdots \lor p_n) = h(p_1) + \cdots + h(p_n) \), \( h(\cdot) \) rank.

The theory of generalized probabilities or belief functions provides a hint about how these relations are connected, for it turns out that structured collections of finite domains (frames) of belief functions have the algebraic structure of locally Birkhoff lattice. By comparing the behavior of $\mathcal{LI}_1, \mathcal{LI}_2$ and $\mathcal{LI}_3$ in the landmark cases of frame and projective lattices (since projective lattices are modular) we inferred a number of general results, proving that for a lattice bounded below $L$:

1. if $L$ is modular then $\mathcal{LI}_2 \Rightarrow \mathcal{LI}_1$;

2. if $L$ is Birkhoff then $\mathcal{LI}_3 \Rightarrow \mathcal{LI}_2$;

3. if $L$ is a modular Birkhoff lattice then $\mathcal{LI}_2 \Rightarrow \mathcal{LI}_3$;

4. if a Birkhoff lattice $L$ is not modular then $\mathcal{LI}_2 \not\equiv \mathcal{LI}_3$.

Therefore we proved an equivalent condition for the modularity of a lattice expressed in terms of equivalence of candidate independence conditions: a Birkhoff lattice bounded below is modular iff $\mathcal{LI}_2 \equiv \mathcal{LI}_3$. 