Lattice modularity and linear independence

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The notion of *abstract linear independence* on lattices (due to Birkhoff) generalizes the important concept of independence of vectors of a linear space. This relation can be expressed in three different equivalent ways for *atoms* of a semimodular lattice bounded below, but are in general distinct when applied to *arbitrary* elements (when they can fail to be independence relations). Let L be a semimodular lattice with initial element, and define $\mathcal{LI}_1, \mathcal{LI}_2, \mathcal{LI}_3$ in the following way:

1. $\{p_1, ..., p_n\} \subset L$ are $\mathcal{LI}_1$-independent if $p_j \not\leq \bigvee_{i \neq j} p_i \;\; \forall j \;\; (\equiv \; p_j \wedge \bigvee_{i \neq j} p_i \neq p_j \;\forall j)$;
2. $\{p_1, ..., p_n\} \subset L$ are $\mathcal{LI}_2$-independent if $p_k \wedge (p_1 \vee \cdots \vee p_{k-1}) = 0, \; k = 2, ..., n$;
3. $\{p_1, ..., p_n\} \subset L$ are $\mathcal{LI}_3$-independent if $h(p_1 \vee \cdots \vee p_n) = h(p_1) + ... + h(p_n)$, $h(\cdot)$ rank.

The theory of generalized probabilities or *belief functions* provides a hint about how these relations are connected, for it turns out that structured collections of finite domains (*frames*) of belief functions have the algebraic structure of locally Birkhoff lattice. By comparing the behavior of $\mathcal{LI}_1, \mathcal{LI}_2$ and $\mathcal{LI}_3$ in the landmark cases of frame and projective lattices (since projective lattices are modular) we inferred a number of general results, proving that for a lattice bounded below L:

1. *if L is modular then* $\mathcal{LI}_2 \Rightarrow \mathcal{LI}_1$;
2. *if L is Birkhoff then* $\mathcal{LI}_3 \Rightarrow \mathcal{LI}_2$;
3. *if L is a modular Birkhoff lattice then* $\mathcal{LI}_2 \Rightarrow \mathcal{LI}_3$;
4. *if a Birkhoff lattice L is* not *modular then* $\mathcal{LI}_2 \not\equiv \mathcal{LI}_3$.

Therefore we proved an equivalent condition for the modularity of a lattice expressed in terms of equivalence of candidate independence conditions: *a Birkhoff lattice bounded below is modular iff* $\mathcal{LI}_2 \equiv \mathcal{LI}_3$.