Geometry of relative plausibility and relative belief of singletons

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Abstract. The study of the interplay between belief and probability can be posed in a geometric framework, in which belief and plausibility functions are represented as points of simplices in a Cartesian space. Probability approximations of belief functions form two homogeneous groups, which we call "affine" and "epistemic" families. In this paper we focus on relative plausibility, belief, and uncertainty of probabilities of singletons, the "epistemic" family. They form a coherent collection of probability transformations in terms of their behavior with respect to Dempster's rule of combination. We investigate here their geometry in both the space of all pseudo belief functions and the probability simplex, and compare it with that of the affine family. We provide sufficient conditions under which probabilities of both families coincide.

Keywords: Theory of evidence, probability transformation, geometric approach, relative plausibility and belief of singletons, relative uncertainty on the probabilities of singletons.

1. Introduction

In the last decades a number of different uncertainty measures (Vakili, 1993) have been proposed as either alternatives to or extensions of classical probability theory. The *theory of evidence* is one the most popular such formalism, extending quite naturally probabilities on finite spaces through the notion of *belief function* (Shafer, 1976). Belief functions $b : 2^\Theta \rightarrow [0, 1]$ (where Θ is a finite set or "frame of discernment", and $2^\Theta = \{A \subseteq \Theta\}$) assign probability values $b(A)$ to *sets* of possibilities $A \subseteq \Theta$ rather than single events. Belief functions assigning probabilities to singletons $x \in \Theta$ only are said "Bayesian", and correspond to probability functions on Θ.
The nexus between belief and probability plays a major role in the theory of evidence (Daniel, 2006), and it is the foundation of a popular approach to evidential reasoning called "transferable belief model" (Smets, 1988). The problem of finding sensible probabilistic and possibilistic (Dubois and Prade, 1990) approximations of belief functions has been widely studied by many authors (Kramosil, 1995; Yaghlane

et al., 2001; Denoeux, 2001; Denoeux and Yaghlane, 2002; Haenni and Lehmann, 2002; Bauer, 1997; Tessem, 1993; Lowrance et al., 1986).

An interesting approach to the problem seeks approximations which enjoy commutativity properties with respect to a specific combination rule, in particular the so-called "Dempster's sum" proposed by G. Shafer (Shafer, 1976), inspired by a series of seminal works by A. Dempster (Dempster, 1968b; Dempster, 1968a). Voorbraak was the first to explore this direction, proposing the adoption of the *relative plausibility of singletons* (Voorbraak, 1989). Given a belief function b with plausibility $pl_b : 2^\Theta \rightarrow [0,1]$, $pl_b(A) = 1 - b(A^c)$, this distribution $\widetilde{pl}_b$ assigns to each element $x \in \Theta$ of the domain Θ its normalized plausibility:

$$\widetilde{pl}_b(x) = \frac{pl_b(x)}{\sum_{y \in \Theta} pl_b(y)}. \tag{1}$$

The relative plausibility of singletons commutes with Dempster's rule of combination $\oplus$, and is therefore seen as consistent with the original semantics of belief functions (Cobb and Shenoy, 2003a; Cobb and Shenoy, 2003b). In addition, $\widetilde{pl}_b$ is a perfect representative of b when combined with other probability measures p: $\widetilde{pl}_b \oplus p = b \oplus p$.

The study of belief functions and their interplay with probability measures can be also posed in a geometric setup (Ha and Haddawy, 1996; Black, 1997). A belief function $b : 2^\Theta \rightarrow [0,1]$ is completely specified by its $2^{|\Theta|} - 2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$ (since $b(\emptyset) = 0$, $b(\Theta) = 1$ for all belief functions). It can then be represented as a point of $\mathbb{R}^{N-2}, N = 2^{|\Theta|}$ (Cuzzolin, 2008b).
In this framework (Cuzzolin, 2007) each belief function is associated with three different geometric entities. These are: the line (b, pl_b) joining b and pl_b, the orthogonal complement $\mathcal{P}^\perp$ of the region of all probability distributions $\mathcal{P}$, and the simplex of probabilities

$$\mathcal{P}[b] = \Big\{ p \in \mathcal{P} : p(A) \geq b(A) \ \forall A \subseteq \Theta \Big\} \tag{2}$$

"consistent" with b. These in turn determine three different probabilities associated with b, i.e.: the intersection probability $p[b]$ (Cuzzolin, 2009a), the orthogonal projection $\pi[b]$ (Cuzzolin, 2007) of b onto $\mathcal{P}$, and the pignistic probability $BetP[b]$ (Smets, 1988) which results from Smets' pignistic transformation (the barycenter $\overline{\mathcal{P}}[b]$ of the simplex $\mathcal{P}[b]$ of consistent probabilities).
If b assigns non-zero mass to focal elements of size 1 or 2 only, then all those probability transformations coincide (Cuzzolin, 2007). The analysis of the simplest binary case, though, suggests that the relative plausibility $\widetilde{pl}_b$ does not fit into this picture.

These two approaches to the probabilistic transformation problem determine a classification of all such transformations of belief functions in two distinct families.
Approximations of the first group ($p[b]$, $\pi[b]$, $BetP[b]$) commute (at least under certain conditions) with affine combination of points in the space of all belief functions (Cuzzolin, 2007). We call this group the *affine* family. On their side, relative plausibility of singletons and *relative* (also called "normalized" (Daniel, 2006)) *belief of singletons*

$$\tilde{b}(x) = \frac{m_b(x)}{\sum_{y\in\Theta} m_b(y)} \tag{3}$$

commute with Dempster's sum (of belief functions and of plausibility functions, respectively). Note that $\tilde{b}$ exists iff b assigns some mass to singletons:

$$\sum_{x\in\Theta} m_b(x) \neq 0. \tag{4}$$

The last two transformations meet a set of dual properties (Cuzzolin, 2008a) with respect to $\oplus$. We call them the *epistemic* family.

1.1. Contributions

The existence of two distinct families of probability transformations of belief functions was originally highlighted by the study of their geometry in the case of a frame of cardinality two. Indeed, the geometry of the affine family in the general case has been thoroughly investigated in (Cuzzolin, 2007). In this paper we focus instead on the epistemic family of probability transformations of belief functions.

In (Cuzzolin, 2008a) it was proven that both relative belief and plausibility meet several properties with respect to Dempster's combination, which can be obtained from each other by means of a form of duality. Here we first give some hints on their rationale, pointing out that they are in fact not consistent with the original belief function. Hence, it is not correct to interpret them as probability "approximations" of b. They instead possess a natural interpretation as best defensive strategies in a non-cooperative game theoretical framework.

We then complete the geometric analysis of all probability transformations by studying the geometry of the epistemic family in the space of all (pseudo) belief functions. Once again, as in the case of the affine family, the geometric features of these functions turn out to possess interpretations in terms of degrees of belief. In particular, their geometry highlights the existence of a third probability distribution measuring the *relative uncertainty* on the probability values of the

singletons (under the evidence represented by b).
As probability transformations map belief functions to probability distributions, it makes sense to analyze their behavior in the probability simplex too. We do this here in the case study of a frame of size 3.

A study of the cross relations between transformations of the affine and epistemic families is paramount to provide us with a complete understanding of the probability transformation problem. This was so far lacking in the nevertheless extensive existing literature. We fill this gap at least in part in the last part of the paper.

1.2. Paper outline

After recalling the basic notions of the theory of belief functions and the related geometric approach (Section 2), we describe the geometry of probability transformation in the binary case (Section 3) which originally hinted at the existence of two families of transformations. In Section 4 we discuss properties and semantics of the epistemic family, providing a rationale for such transformations, and suggesting a more suitable interpretation for them in a game theory setup.
We then move in Section 5 to study the geometry of the pair $\widetilde{pl_b}, \tilde{b}$ in the general case. Such geometry turns out to be a function of another probability $R[b]$, which measures the relative uncertainty on the probability values of each singleton. Several examples will illustrate the relation between the geometry of the involved functions and their properties in terms of degrees of belief. Their geometry is also investigated in the probability simplex (Section 6), at least in the case study of a frame of size 3.
Finally, as a step towards a complete understanding of the probability transformation problem, we discuss (Section 7) the relationship between the affine and epistemic families of probability transformations. Abstracting from the binary case study we provide sufficient conditions under which they all coincide, in terms of equal distribution of masses and equal contribution to the plausibility of the singletons.

2. A geometric approach to the theory of evidence

Belief and plausibility functions. A *basic probability assignment* or *basic belief assignment* over a finite set Θ (frame of discernment) is a function $m : 2^\Theta \rightarrow [0,1]$ such that $m(\emptyset) = 0$, $\sum_{A\subseteq\Theta} m(A) = 1$, $m(A) \geq 0 \; \forall A \subseteq \Theta$. The *belief function* $b : 2^\Theta \rightarrow [0,1]$ associated with m is given by $b(A) = \sum_{B\subseteq A} m(B)$. They are linked by the *Moebius inversion formula* $m_b(A) = \sum_{B\subseteq A} (-1)^{|A-B|} b(B)$.

A *Bayesian* belief function is a special belief function which assigns mass to singletons only: $m_b(A) = 0, |A| > 1$. Each Bayesian belief function b is in 1-1 correspondence with the probability distribution $p : \Theta \to [0,1]$ such that $p(x) = m_b(x)$. A dual representation of the evidence encoded by a belief function b is the *plausibility function* $pl_b : 2^\Theta \to [0,1]$. Its value $pl_b(A) = 1 - b(A^c) = \sum_{B\cap A\neq\emptyset} m_b(B)$ for each event A expresses the amount of evidence *not against* A.
Two or more belief functions can be combined by Dempster's sum.

Definition 1. *The* orthogonal sum *or* Dempster's sum *of two belief functions* $b_1, b_2 : 2^\Theta \to [0,1]$ *is a new belief function* $b_1 \oplus b_2 : 2^\Theta \to [0,1]$ *with mass*

$$m_{b_1\oplus b_2}(A) = \frac{\sum_{B\cap C=A} m_{b_1}(B)\; m_{b_2}(C)}{\sum_{B\cap C\neq\emptyset} m_{b_1}(B)\; m_{b_2}(C)} \quad \forall A \subseteq \Theta, \tag{5}$$

where m_{b_i} *denotes the basic belief assignment associated with* b_i.

Belief and plausibility spaces. Given a frame of discernment Θ, a belief function $b : 2^\Theta \to [0,1]$ is completely specified by its $N-2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$, $N = 2^{|\Theta|}$, and can be represented as a vector with $N-2$ entries, i.e., a point of $\mathbb{R}^{N-2}$. We call *belief space* $\mathcal{B}$ the set of points of $\mathbb{R}^{N-2}$ which correspond to a belief function. It is not difficult to prove (Cuzzolin, 2008b) that $\mathcal{B}$ *is convex.*
Let us denote by b_A the *categorical* (Smets and Kennes, 1994) belief function assigning all the mass to a single subset $A \subseteq \Theta$: $m_{b_A}(A) = 1$, $m_{b_A}(B) = 0$ for all $B \neq A$. The belief space $\mathcal{B}$ is a *simplex*, the convex closure[1] Cl of all categorical belief functions $\{b_A\}$:

$$\mathcal{B} = Cl(b_A,\ \emptyset \subsetneq A \subseteq \Theta). \tag{7}$$

Vectors $v_1, ..., v_n$ of a linear space $\mathbb{R}^d$ can be linearly combined in order to yield new vectors, for instance $v' = \alpha_1 v_1 + ... + \alpha_n v_n$. This holds for belief functions too, when seen as vectors of $\mathbb{R}^{N-2}$. Each belief function $b \in \mathcal{B}$ can then be written as a convex sum of the vectors b_A representing the categorical belief functions as

$$b = \sum_{\emptyset\subsetneq A\subseteq\Theta} m_b(A) \cdot b_A, \tag{8}$$

which scalar coefficients $m_b(A) \in [0,1]$.
The set $\mathcal{P}$ of all the Bayesian belief functions on Θ is the simplex

[1] Here Cl denotes the convex closure operator:

$$Cl(b_1, ..., b_k) = \Big\{ b \in \mathcal{B} : b = \alpha_1 b_1 + \cdots + \alpha_k b_k, \sum_i \alpha_i = 1,\ \alpha_i \geq 0\ \forall i \Big\} \tag{6}$$

where $b_1, ..., b_k$ is an arbitrary collection of belief functions.

determined by all the categorical functions associated with singletons[2]: $\mathcal{P} = Cl(b_x, x \in \Theta)$. The simplex $\mathcal{P}$ is naturally in correspondence with the simplex of all probability distributions on Θ. Analogously, we can call *plausibility space* the region $\mathcal{PL}$ of $\mathbb{R}^{N-2}$ whose points correspond to plausibility functions, i.e., they are vectors of the form

$$pl_b = [pl_b(A), \emptyset \subsetneq A \subsetneq \Theta]'.$$

This is also a simplex $\mathcal{PL} = Cl(pl_A, \emptyset \subsetneq A \subseteq \Theta)$, whose vertices are the vectors (Cuzzolin, 2003)

$$pl_A = -\sum_{B \subseteq A} (-1)^{|B|} b_B. \tag{9}$$

The A-th vertex pl_A of the plausibility space is the plausibility vector associated with the categorical belief function b_A: $pl_A = pl_{b_A}$.
Similarly to the case of belief functions (8), each plausibility function pl_b (as a vector of $\mathbb{R}^{N-2}$) can be uniquely expressed as convex combination of all categorical plausibility functions (9) as

$$pl_b = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) \cdot pl_A. \tag{10}$$

The "vacuous" belief function for which $b_\Theta(\Theta) = 1$, $b_\Theta(A) = 0$ for $A \neq \Theta$ is the origin $\mathbf{0}$ of $\mathbb{R}^{N-2}$. Its plausibility vector $pl_\Theta = \mathbf{1}$ is the vector whose entries are all equal to 1.

Pseudo belief functions. The basic belief assignment m_b associated with a belief function b meets the positivity axiom: $m_b(A) \geq 0$ for all $A \subseteq \Theta$. Sum functions of the same form of belief functions

$$\varsigma(A) = \sum_{B \subseteq A} m_\varsigma(B)$$

whose Moebius transform m_ς meets the normalization axiom, $\varsigma(\Theta) = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_\varsigma(A) = 1$, but is not necessarily non-negative are called *pseudo belief functions* (Smets, 1995). Notice that this time $m_\varsigma(A) \in \mathbb{R}$ for all $A \subseteq \Theta$ in general. Geometrically, they fill the entire $\mathbb{R}^{N-2}$: each vector of $\mathbb{R}^{N-2}$ corresponds to a pseudo belief function.

[2] With a slight abuse of notation we will denote by $m(x)$, $b(x)$, $pl_b(x)$ etc. instead of $m(\{x\}), b(\{x\}), pl_b(\{x\})$ the values of the set functions of interest on a singleton.

3. A geometric interplay of belief and probability

Figure 1 shows the geometry of belief $\mathcal{B}$ and plausibility $\mathcal{PL}$ spaces for a frame $\Theta_2 = \{x, y\}$ of cardinality 2. Belief and plausibility vectors are, in this case, points of a plane with coordinates

$$\begin{array}{rcl} b & = & [b(x) = m_b(x), b(y) = m_b(y)]' \\ pl_b & = & [pl_b(x) = 1 - m_b(y), pl_b(y) = 1 - m_b(x)]' \end{array}$$

respectively. These two simplices

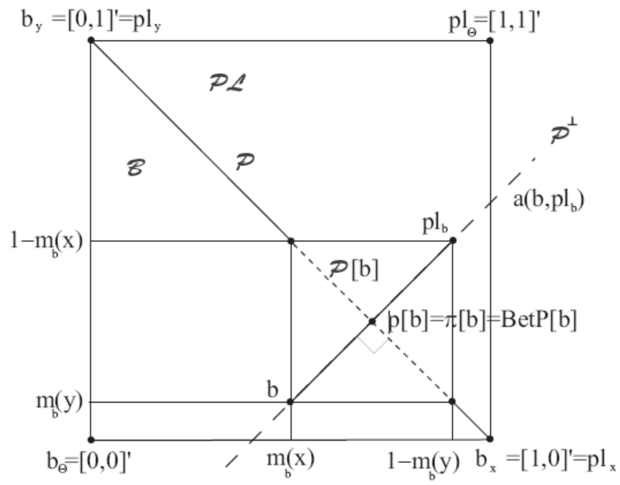

Figure 1. Geometry of belief and plausibility functions in the case of a frame of cardinality 2: $\Theta_2 = \{x, y\}$. The probabilities generated by transformations of the affine family (pignistic function $BetP[b]$, orthogonal projection $\pi[b]$ of b onto $\mathcal{P}$, and intersection $p[b]$ of the segment $Cl(b, pl_b)$ with $\mathcal{P}$) all coincide. The segment $\mathcal{P}[b]$ of all the probabilities consistent with b is denoted by a dotted line. The orthogonal complement $\mathcal{P}^{\perp}$ of $\mathcal{P}$ in $p[b] = BetP[b]$ is shown as a dashed line.

$$\mathcal{B} = Cl(b_\Theta = \mathbf{0}, b_x, b_y), \qquad \mathcal{PL} = Cl(pl_\Theta = \mathbf{1}, pl_x = b_x, pl_y = b_y)$$

are symmetric with respect to the probability simplex $\mathcal{P} = Cl(b_x, b_y)$. Each pair (b, pl_b) determines a line orthogonal to $\mathcal{P}$. On this line b and pl_b lay on symmetric positions on the two sides of the Bayesian simplex. This fact was also noticed, at least for two-element frames of discernment, in (Daniel, 2006).

3.1. Two families of probability transformations

The binary example pictorially illustrates the different probability transformations of a belief function b.

The affine family. Each belief function b is geometrically associated with three loci (Figure 1): the line $a(b, pl_b)$[3] joining b and pl_b, the set of probabilities $\mathcal{P}[b]$ consistent with b (2), and the orthogonal complement $\mathcal{P}^\perp$ of $\mathcal{P}$. The line $a(b, pl_b)$ is always orthogonal to $\mathcal{P}$ (Cuzzolin, 2007). It is also associated with a probability distribution called *intersection probability*

$$p[b](x) = b(x) + \beta[b](pl_b(x) - b(x)) \tag{11}$$

where

$$\beta[b] = \frac{1 - \sum_{x\in\Theta} b(x)}{\sum_{x\in\Theta}(pl_b(x) - b(x))} \in [0,1] \tag{12}$$

is a scalar function of b.

The intersection probability is in general distinct from the *orthogonal projection* $\pi[b]$ of b onto $\mathcal{P}$ (Cuzzolin, 2007), and the *pignistic probability*

$$BetP[b](x) = \sum_{A\supseteq\{x\}} \frac{m_b(A)}{|A|}, \tag{13}$$

geometrically the barycenter of the region $\mathcal{P}[b]$ of consistent probabilities (Chateauneuf and Jaffray, 1989). In the binary case all those probability transformations of b *coincide*: $\pi[b] = BetP[b] = p[b]$ (see Figure 1).

The epistemic family. Figure 2 shows though that the relative plausibility of singletons (1) does not coincide with the other transformations even in the binary case. Indeed

$$\begin{aligned} \widetilde{pl}_b &= [\widetilde{pl}_b(x), \widetilde{pl}_b(y)]' = \frac{1}{pl_b(x) + pl_b(y)}[pl_b(x), pl_b(y)]' \\ &= \tfrac{1}{pl_b(x)+pl_b(y)} pl_b + \Big(1 - \tfrac{1}{pl_b(x)+pl_b(y)}\Big)\mathbf{0} \in a(pl_b, \mathbf{0}), \end{aligned}$$

so that $\widetilde{pl}_b$ is simply the intersection of the Bayesian segment $\mathcal{P}$ with the line $a(pl_b, \mathbf{0})$ joining pl_b and the origin of $\mathbb{R}^2$ (the vacuous belief function $b_\Theta = \mathbf{0}$).

[3] An *affine combination* of k points $v_1, ..., v_k \in \mathbb{R}^m$ is a sum $\alpha_1 v_1 + \cdots + \alpha_k v_k$ whose coefficients sum to one: $\sum_i \alpha_i = 1$. We will denote by $a(v_1, .., v_k)$ the affine subspace of $\mathbb{R}^m$ generated by the points $v_1, ..., v_k \in \mathbb{R}^m$, i.e., the set $\{v \in \mathbb{R}^m : v = \alpha_1 v_1 + \cdots + \alpha_k v_k, \sum_i \alpha_i = 1\}$. Note that convex combination is a special case of affine combination, in turn a special case of linear combination (Smets, 2005).

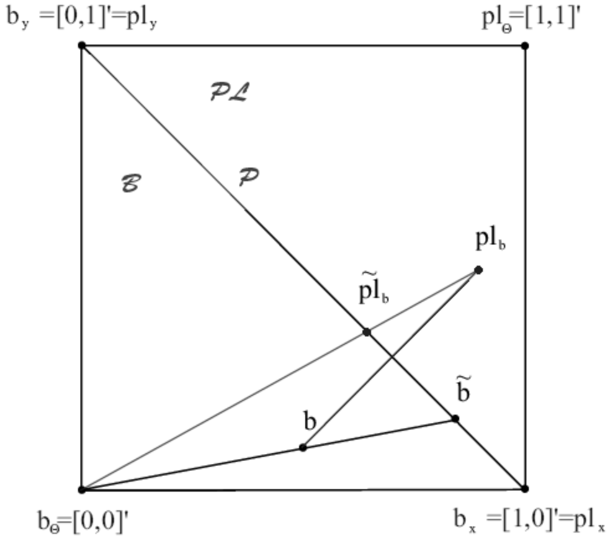

Figure 2. Geometry of the epistemic family of probability transformations, relative belief $\tilde{b}$ and plausibility $\widetilde{pl_b}$ of singletons, in the binary case.

The relative belief of singletons (3) is the intersection of the line $a(b, b_\Theta)$ joining b and b_Θ with the probability simplex, as

$$\tilde{b}(x) = \frac{m_b(x)}{1 - m_b(\Theta)}, \quad \tilde{b}(y) = \frac{m_b(y)}{1 - m_b(\Theta)}.$$

A comparison of Figure 1 and 2 suffices to prove that relative belief and plausibility do not belong to the same geometric group of probability transformations. This is a reflection of the fact that they relate to different operators in the space of belief functions $\mathcal{B}$ (Smets, 2005; Cuzzolin, 2007).

4. Semantics and properties of the epistemic family

4.1. Affine and epistemic families

Proposition 1. *Both orthogonal transformation $\pi[b] : \mathcal{B} \rightarrow \mathcal{P}$ and pignistic transformation $BetP[b] : \mathcal{B} \rightarrow \mathcal{P}$ (mapping any b.f. b onto (11) and (13), respectively) commute with respect to affine combination, i.e., $\pi[\alpha_1 b_1 + \alpha_2 b_2] = \alpha_1 \pi[b_1] + \alpha_2 \pi[b_2]$, $BetP[\alpha_1 b_1 + \alpha_2 b_2] = \alpha_1 BetP[b_1] + \alpha_2 BetP[b_2]$ whenever $\alpha_1 + \alpha_2 = 1$ $\forall b_1, b_2 \in \mathcal{B}$.*

The intersection probability $p[b]$ also commutes (under certain conditions) with affine combination. It is natural to call the group $BetP[b]$,

$\pi[b]$, $p[b]$ the "affine" family of probability transformations of b.
On the other side, both $\widetilde{pl}_b$ (Voorbraak, 1989) and $\tilde{b}$ (Cuzzolin, 2009b) commute with respect to Dempster's rule of combination of belief or plausibility functions, respectively. Let us see why in greater detail.

Dempster's rule can be naturally extended to pseudo belief functions (Cuzzolin, 2009b) by applying Dempster's rule (5) to the Moebius inverses m_{ς_1}, m_{ς_2} of any pair of pseudo belief functions ς_1, ς_2. We can still denote the orthogonal sum of two pseudo belief functions ς_1, ς_2 by $\varsigma_1 \oplus \varsigma_2$. As plausibility functions (in the normalized case) are such that $pl_b(\Theta) = 1$, they are also pseudo belief functions. Formally they are "normalized sum functions" $pl_b(A) = \sum_{B \subseteq A} \mu_b(B)$, with Moebius transform μ_b such that $\sum_{B \subseteq \Theta} \mu_b(B) = 1$. We call μ_b the "basic plausibility assignment" associated with b and pl_b (Cuzzolin, 2009c). The basic plausibility assignment is such that $m_b(x) = \sum_{A \supseteq \{x\}} \mu_b(A)$.
Given two plausibility functions pl_{b_1} and pl_{b_2}, Dempster's rule can therefore be applied to their Moebius inverses μ_{b_1} and μ_{b_2}, yielding a Dempster's sum $pl_{b_1} \oplus pl_{b_2}$ of plausibility functions (Cuzzolin, 2009c).

Let us then introduce the *relative plausibility operator* and the *relative belief operator*

$$\begin{array}{rcl} \tilde{pl} : \mathcal{B} & \rightarrow & \mathcal{P} \\ b & \mapsto & \tilde{pl}[b] = \widetilde{pl}_b \end{array} \qquad \begin{array}{rcl} \tilde{b} : \mathcal{PL} & \rightarrow & \mathcal{P} \\ pl_b & \mapsto & \tilde{b}[pl_b] = \tilde{b} \end{array} \tag{14}$$

where

$$\begin{array}{rcl} \tilde{pl}_b[b](x) & = & \dfrac{\sum_{A \supseteq x} m_b(A)}{\sum_{y \in \Theta} \sum_{A \supseteq y} m_b(A)} = \dfrac{pl_b(x)}{\sum_{y \in \Theta} pl_b(y)} = \tilde{pl}_b(x) \quad \forall x \in \Theta \\ \tilde{b}[pl_b](x) & = & \dfrac{\sum_{A \supseteq x} \mu_b(A)}{\sum_{y \in \Theta} \sum_{A \supseteq y} \mu_b(A)} = \dfrac{m_b(x)}{\sum_{y \in \Theta} m_b(y)} = \tilde{b}(x) \quad \forall x \in \Theta \end{array} \tag{15}$$

in virtue of the cited property of basic plausibility assignments.
From Equation (15) it is apparent that the relative belief of singletons can be thought of indifferently as an operator on belief functions with domain $\mathcal{B}$ or on plausibility functions with domain $\mathcal{PL}$ (as belief functions and plausibility functions carry the same evidence). We wrote it here in its second form as this is instrumental to prove the following result.

Proposition 2. *The relative plausibility operator commutes with Dempster's combination* of belief functions. *The relative belief operator commutes with Dempster's combination* of plausibility functions. *Namely*

$$\tilde{pl}[b_1 \oplus b_2] = \tilde{pl}[b_1] \oplus \tilde{pl}[b_2]; \qquad \tilde{b}[pl_1 \oplus pl_2] = \tilde{b}[pl_1] \oplus \tilde{b}[pl_2],$$

where $\oplus$ is used to denote a Dempster's sum of Bayesian belief functions on the right hand side of each equality.

The proof can be found in (Cuzzolin, 2009b). In fact, $\widetilde{pl}_b$ and $\tilde{b}$ meet a number of dual properties (Cobb and Shenoy, 2003a; Cuzzolin, 2009b) obtained by swapping the role of belief and plausibility. This is true in particular for Voorbraak's representation theorem.

Given the central role of Dempster's rule in the Theory of Evidence it is natural to call the group $\{\widetilde{pl}_b, \tilde{b}\}$ the "epistemic" family of probability transformations. The different geometric behavior of the two groups of transformations is nothing but the reflection of a deeper intrinsic diversity of their semantics. The geometry of the family of affine probability transformations has been studied in (Cuzzolin, 2007). In this paper we will focus on the epistemic family.

4.2. Rationale of the epistemic family of transformations

4.2.1. *Two dual scenarios*

An insight on the meaning of $\tilde{b}$ and $\widetilde{pl}_b$ comes from the interpretation of belief functions as constraints on the actual allocation of mass of an underlying unknown probability distribution on Θ. According to this interpretation, the mass $m_b(A)$ assigned to each focal element $A \subseteq \Theta$ can float freely among its elements $x \in A$. A probability distribution "consistent" with b emerges by arbitrarily redistributing the mass of each focal element to its singletons. The set of all and only such consistent probabilities is (2). In this framework, the relative plausibility of singletons $\widetilde{pl}_b$ (1) can be interpreted as follows:

1. for each singleton $x \in \Theta$ the most *optimistic* hypothesis in which the mass of all $A \supseteq \{x\}$ focuses on x is considered, yielding $\{pl_b(x), x \in \Theta\}$;

2. this assumption, however, is not consistent with the evidence at hand as it is supposed to hold for all singletons (many of which belong to the same higher-size focal elements);

3. nevertheless, the obtained plausibility values $pl_b(x)$ are normalized to yield an admissible probability.

The probability transformation $\widetilde{pl}_b$ is associated with the (incoherent) scenario in which all the mass that can be assigned to a singleton is actually assigned to it, and this for all singletons.

Similarly in the case of the relative belief of singletons (3):

1'. for each singleton $x \in \Theta$ the most *pessimistic* hypothesis in which only the mass of $\{x\}$ itself actually focuses on x is taken into account, yielding $\{b(x) = m_b(x), x \in \Theta\}$;

2'. this assumption is also inconsistent with the evidence given by b, as the mass of all higher-size focal elements is *not* assigned to *any* singletons;

3'. the obtained values $b(x)$ are again normalized to produce a valid probability.

Dually, $\tilde{b}$ reflects the (still not coherent) choice of assigning to x only the mass that the b.f. b (seen as a constraint) assures it belong to x.

4.2.2. *Inconsistency of relative belief and plausibility of singletons*
The fact that both relative belief and plausibility of singletons emerge from incoherent scenarios in terms of the available evidence is reflected by the fact that none of them are guaranteed to belong to the set of probabilities (2) consistent with b (Cuzzolin, 2009b).

Proposition 3. *Relative belief and plausibility of singletons are not always consistent.*

It is easy to find a counterexample for both $\widetilde{pl_b}$ and $\tilde{b}$. However, it should be noticed that both transformations are consistent for quasi-Bayesian b.f.s, i.e., belief functions b such that $m_b(A) = 0$ for $1 < |A| < |\Theta|$).

In this sense, then, it is probably not correct to refer to relative belief and plausibility of singletons as probability "approximations" of a belief function b, as they are not necessary one of the (consistent) probability distributions (2) of the class $\mathcal{P}[b]$ represented by b.

4.2.3. *A game/utility theory interpretation*
The most interesting interpretation of those two probability transformations is provided by game/utility theory.
Suppose there exists a utility function $u : \Theta \to \mathbb{R}^+$ which measures the relative satisfaction (for us) of the different outcomes $x \in \Theta$. The belief value of each singleton $x \in \Theta$ measures the minimal support x can receive from a distribution $p \in \mathcal{P}[b]$ of the family (2) associated with the belief function b:

$$b(x) = \min_{p \in \mathcal{P}[b]} p(x).$$

Hence $x_{maximin} = \arg\max_{x\in\Theta} b(x)$ is the outcome which maximizes such minimal support.
When we normalize to obtain the relative belief of singletons, this outcome is conserved:

$$x_{maximin} = \arg\max_{x\in\Theta} \tilde{b}(x) = \arg\max_{x\in\Theta} \min_{p\in\mathcal{P}[b]} p(x).$$

Consider now a game theory scenario in which our opponent has the first move and is free to choose the probability distribution $p \in \mathcal{P}[b]$ consistent with b in order to damage us. If the utility function is constant, i.e., no element of Θ can be preferred over the others, $x_{maximin}$ (the peak(s) of the relative belief of singletons) represents the best possible defensive strategy aimed at maximizing the minimal utility of the possible outcomes.

Dually, $pl_b(x)$ measures the maximal possible support to x by a distribution consistent with b, so that

$$x_{minimax} = \arg\min_{x\in\Theta} \widetilde{pl}_b(x) = \arg\min_{x\in\Theta} \max_{p\in\mathcal{P}[b]} p(x)$$

is the outcome which minimizes the maximal possible support.
Suppose for sake of simplicity that the loss function $l = 1-u : \Theta \rightarrow \mathbb{R}^+$ which measures the relative dissatisfaction of the outcomes is constant, and that in the same game theory setup our opponent is (again) free to pick a consistent probability distribution $p \in \mathcal{P}[b]$. Then the element with minimal relative plausibility is the best possible defensive strategy aimed at minimizing the maximum possible loss.

Note that when the utility function is *not* constant the above minimax and maximin problems naturally generalize as

$$\begin{array}{rcl} x_{maximin} & = & \arg\max_{x\in\Theta} \tilde{b}(x)u(x), \\ x_{minimax} & = & \arg\min_{x\in\Theta} \widetilde{pl}_b(x)(1-u(x)). \end{array}$$

Relative belief and plausibility of singletons still play the crucial role in determining the best strategy.

5. Geometry in the space of pseudo belief functions

As stated above, the aim of this paper is to study the geometry of the epistemic family. In Section 5 we analyze the geometry of the pair $\widetilde{pl}_b, \tilde{b}$ in the space of all pseudo belief functions. This geometry can be reduced to that of two specific pseudo belief functions called "plausibility of singletons" (16) and "belief of singletons" (18), introduced in Sections (5.1) and (5.2). We point out how their geometry can be described in terms of three planes (5.3) and angles (5.4) in the belief space. Such angles are in turn related to a probability $R[b]$ which measures the *relative uncertainty on the probabilities of singletons* determined by b, and can be assimilated to $\tilde{b}$ and $\widetilde{pl}_b$ as the third member of the epistemic family of transformations. As $\tilde{b}$ does not exist when $\sum_x m_b(x) = 0$, this singular case needs to be discussed separately (Section 5.5).

Probability transformations, however, map belief functions onto probability distributions. It makes then sense to study their behavior in the simplex of all probabilities. We get some insight on this in Section 6 in the case study of a size-three frame.

We close the paper (Section 7) by discussing the relationship between the two families of probability transformations. The proofs of

Theorems 2-5, 8, Lemma 1 and Corollaries 1 and 2 can be found in the Appendix.

5.1. Plausibility of singletons and relative plausibility

Let us introduce the scalar quantity $k_{pl_b} = \sum_{x\in\Theta} pl_b(x)$ as a measure of the total plausibility of singletons. Let us also call *plausibility of singletons* the pseudo belief function $\overline{pl}_b : 2^\Theta \to [0,1]$ with Moebius transform $m_{\overline{pl}_b} : 2^\Theta \to \mathbb{R}$ given by

$$m_{\overline{pl}_b}(x) = pl_b(x)\ \forall x \in \Theta, \qquad m_{\overline{pl}_b}(\Theta) = 1 - \sum_x pl_b(x) = 1 - k_{pl_b},$$
$$m_{\overline{pl}_b}(A) = 0 \quad \forall A \subseteq \Theta : |A| \neq 1, n.$$

Indeed $m_{\overline{pl}_b}$ meets the normalization constraint

$$\sum_{A\subseteq\Theta} m_{\overline{pl}_b}(A) = \sum_{x\in\Theta} pl_b(x) + \Big(1 - \sum_{x\in\Theta} pl_b(x)\Big) = 1.$$

Then, as $1 - k_{pl_b} \leq 0$, $\overline{pl}_b$ is a pseudo belief function (Section 2). Note that $\overline{pl}_b$, however, is *not* a plausibility function.

In the belief space $\overline{pl}_b$ is represented by the vector

$$\overline{pl}_b = \sum_{x\in\Theta} pl_b(x)\ b_x + (1 - k_{pl_b})\ b_\Theta = \sum_{x\in\Theta} pl_b(x)\ b_x, \tag{16}$$

for (as usual) $b_\Theta = \mathbf{0}$ is the origin of the reference frame in $\mathbb{R}^{N-2}$.

Theorem 1. *$\widetilde{pl}_b$ is the intersection of the line joining vacuous belief function b_Θ and plausibility of singletons $\overline{pl}_b$ with the probability simplex.*

Proof. By Equations (1) (16) we have that $\widetilde{pl}_b = \sum_{x\in\Theta} \widetilde{pl}_b(x) b_x = \overline{pl}_b / k_{pl_b}$. Since $b_\Theta = \mathbf{0}$ is the origin of the reference frame, $\widetilde{pl}_b$ lies on the segment $Cl(\overline{pl}_b, b_\Theta)$. This in turn implies $\widetilde{pl}_b = Cl(\overline{pl}_b, b_\Theta) \cap \mathcal{P}$. □

This result, at least in the binary case, appeared in (Daniel, 2006) too. The geometry of $\widetilde{pl}_b$ depends on that of $\overline{pl}_b$ through Theorem 1. In the binary case $\overline{pl}_b = pl_b$, and we go back to the situation of Figure 2.

5.2. Belief of singletons and relative belief

In Section 3.1 we introduced the intersection probability as a member of the affine family. By definition of $p[b]$ (11) it follows that

$$\begin{aligned} p[b] &= \sum_{x\in\Theta} m_b(x) b_x + \beta[b] \sum_{x\in\Theta} (pl_b(x) - m_b(x)) b_x \\ &= (1 - \beta[b]) \sum_{x\in\Theta} m_b(x) b_x + \beta[b] \sum_{x\in\Theta} pl_b(x) b_x. \end{aligned} \tag{17}$$

Analogously to what done for the plausibility of singletons, we can define the belief function (*belief of singletons*) $\overline{b} : 2^\Theta \rightarrow [0,1]$ with basic belief assignment

$$m_{\overline{b}}(x) = m_b(x), \quad m_{\overline{b}}(\Theta) = 1 - k_{m_b}, \quad m_{\overline{b}}(A) = 0 \quad \forall A \subsetneq \Theta : |A| \neq 1, n,$$

where the scalar quantity $k_{m_b} = \sum_{x\in\Theta} m_b(x)$ measures the total mass of singletons. The belief of singletons assigns to Θ all the mass b gives to non-singletons. In the belief space $\overline{b}$ is represented by the vector

$$\overline{b} = \sum_{x\in\Theta} m_b(x) b_x + (1 - k_{m_b}) b_\Theta = \sum_{x\in\Theta} m_b(x) b_x \tag{18}$$

(as again $b_\Theta = \mathbf{0}$). Equation (17) can then be written as

$$p[b] = (1 - \beta[b])\, \overline{b} + \beta[b]\, \overline{pl}_b. \tag{19}$$

In the binary case $\overline{b} = b$ and $\overline{pl}_b = pl_b$, so that the plausibility of singletons is a plausibility function (Figure 2).

5.3. A three plane geometry

The geometry of relative plausibility and belief of singletons can therefore be reduced to that of the pair $\overline{pl}_b$, $\overline{b}$.
According to Equations (8) and (10), a belief function b and the corresponding plausibility function pl_b have the same coordinates with respect to the vertices of belief b_A and plausibility pl_A spaces:

$$b = \sum_{\emptyset \neq A \subseteq \Theta} m_b(A) b_A \quad \leftrightarrow \quad pl_b = \sum_{\emptyset \neq A \subseteq \Theta} m_b(A) pl_A.$$

They form a pair of "dual" vectors in the respective spaces.
In the same way, plausibility $\overline{pl}_b$ and belief $\overline{b}$ of singletons have duals (that we can denote by $\widehat{pl}_b$ and $\widehat{b}$) which have the same coordinates in the plausibility space: $\overline{b} \leftrightarrow \widehat{b}$, $\overline{pl}_b \leftrightarrow \widehat{pl}_b$. They can be written as

$$\begin{aligned} \widehat{b} &= \sum_{x\in\Theta} m_b(x) pl_x + (1 - k_{m_b}) pl_\Theta = \overline{b} + (1 - k_{m_b}) pl_\Theta \\ \widehat{pl}_b &= \sum_{x\in\Theta} pl_b(x) pl_x + (1 - k_{pl_b}) pl_\Theta = \overline{pl}_b + (1 - k_{pl_b}) pl_\Theta \end{aligned} \tag{20}$$

(as $pl_x = b_x$ for all $x \in \Theta$), where, again, $pl_\Theta = \mathbf{1}$. We can prove that (see Appendix):

Theorem 2. *The line passing through the duals (20) of plausibility of singletons (16) and belief of singletons (18) crosses* $p[b]$ *too, and*

$$\beta[b]\, (\widehat{pl}_b - \widehat{b}) + \widehat{b} = p[b] = \beta[b]\, (\overline{pl}_b - \overline{b}) + \overline{b}. \tag{21}$$

If $k_{m_b} \neq 0$ the geometry of relative plausibility and belief of singletons can therefore be described in terms of the *three planes*

$$a(\overline{pl}_b, p[b], \widehat{pl}_b), \quad a(b_\Theta, \widetilde{pl}_b, pl_\Theta), \quad a(b_\Theta, \tilde{b}, pl_\Theta)$$

(see Figure 3), where $\tilde{b} = \overline{b}/k_{m_b}$ is the relative belief of singletons. Namely:

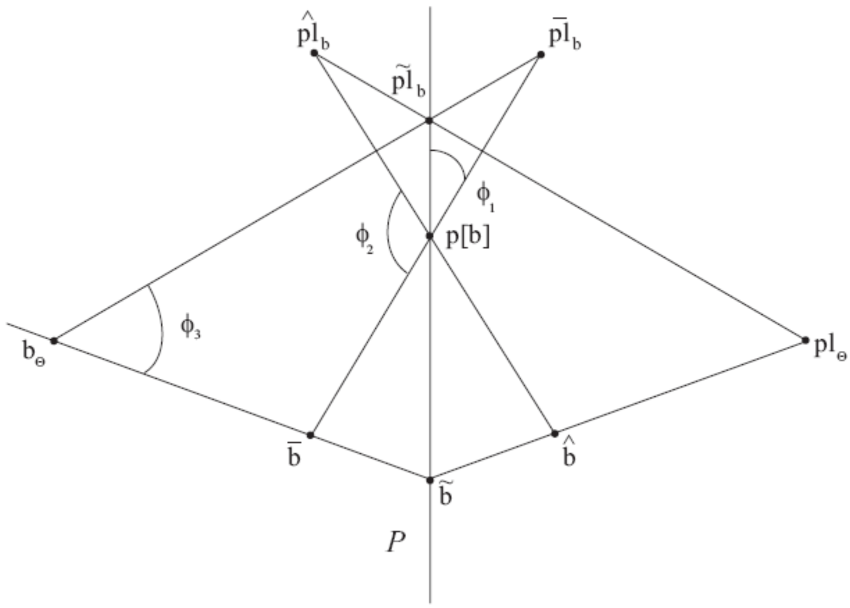

Figure 3. Planes and angles describing the geometry of relative plausibility and belief of singletons, in terms of plausibility of singletons $\overline{pl}_b$ and belief of singletons $\overline{b}$. Geometrically two lines or three points are sufficient to uniquely determine a plane passing through them. The two lines $a(\overline{b}, \overline{pl}_b)$ and $a(\widehat{b}, \widehat{pl}_b)$ uniquely determine a plane $a(\overline{b}, p[b], \widehat{b})$. Two other planes are uniquely determined by the origins of belief b_Θ and plausibility pl_Θ spaces together with either the relative plausibility of singletons $\widetilde{pl}_b$ or the relative belief of singletons $\tilde{b}$: $a(b_\Theta, \widetilde{pl}_b, pl_\Theta)$ (top of the diagram) and $a(b_\Theta, \tilde{b}, pl_\Theta)$ (bottom), respectively. The angles $\phi_1[b], \phi_2[b], \phi_3[b]$ are all independent, as the value of each of them reflects a different property of the original belief function b. The original belief b and plausibility pl_b functions do not appear here to simplify the geometric interpretation of $\tilde{b}, \widetilde{pl}_b$. They play a role only through the related plausibility of singletons (16) and belief of singletons (18).

1. $p[b]$ is the intersection of $a(\overline{b}, \overline{pl}_b)$ and $a(\widehat{b}, \widehat{pl}_b)$, and has the same affine coordinate on the two lines (Section 5.1). Those two lines then span a plane which we can denote by:
$$a(\overline{b}, p[b], \widehat{b}) = a(\overline{pl}_b, p[b], \widehat{pl}_b).$$

2. Furthermore, by definition,

$$\widetilde{pl}_b - b_\Theta = (\overline{pl}_b - b_\Theta)/k_{pl_b} \tag{22}$$

while (20) implies $\widetilde{pl}_b = \overline{pl}_b/k_{pl_b} = [\widehat{pl}_b - (1-k_{pl_b})pl_\Theta]/k_{pl_b}$ so that

$$\widetilde{pl}_b - pl_\Theta = (\widehat{pl}_b - pl_\Theta)/k_{pl_b}. \tag{23}$$

By comparing (22) and (23) we realize that $\widetilde{pl}_b$ has the same affine coordinate on the two lines $a(b_\Theta, \overline{pl}_b)$ and $a(pl_\Theta, \widehat{pl}_b)$, which intersect exactly in $\widetilde{pl}_b$. The functions $b_\Theta, pl_\Theta, \widetilde{pl}_b, \overline{pl}_b$ and $\widehat{pl}_b$ therefore determine another plane which we can denote by:

$$a(b_\Theta, \widetilde{pl}_b, pl_\Theta).$$

3. Analogously, by definition, $\tilde{b} - b_\Theta = (\bar{b} - b_\Theta)/k_{m_b}$ while (20) yields $\tilde{b} - pl_\Theta = (\hat{b} - pl_\Theta)/k_{m_b}$. The relative belief of singletons then has the same affine coordinates on the two lines $a(b_\Theta, \bar{b})$ and $a(pl_\Theta, \hat{b})$. The latter intersect exactly in $\tilde{b}$. The quantities $b_\Theta, pl_\Theta, \tilde{b}, \bar{b}$ and $\hat{b}$ therefore determine a single plane denoted by

$$a(b_\Theta, \tilde{b}, pl_\Theta).$$

5.4. A geometry of three angles

In the binary case, $b = \bar{b} = \widehat{pl}_b = [m_b(x), m_b(y)]'$, $pl_b = \overline{pl}_b = \hat{b} = [1 - m_b(y), 1 - m_b(x)]'$ and all these quantities are coplanar. This suggests a description of the geometry of $\widetilde{pl}_b, \tilde{b}$ in terms of the three angles

$$\phi_1[b] = \widehat{\widetilde{pl}_b\, p[b]\, \overline{pl}_b},\ \phi_2[b] = \widehat{\bar{b}\, p[b]\, \widehat{pl}_b},\ \phi_3[b] = \widehat{\tilde{b}\, b_\Theta\, \widetilde{pl}_b} \tag{24}$$

(Figure 3). Those angles are all independent from each other, and each of them has a distinct interpretation in terms of degrees of belief. Different values of those angles reflect different properties of the belief function b and the associated probability transformations.

Orthogonality condition for $\phi_1[b]$ and relative uncertainty. The line $a(b, pl_b)$ is always orthogonal to $\mathcal{P}$. The line $a(\bar{b}, \overline{pl}_b)$, though, is *not* in general orthogonal to the probabilistic subspace. Let us see why.
The simplex $\mathcal{P} = Cl(b_x, x \in \Theta)$ determines an affine (or vector) space $a(\mathcal{P}) = a(b_x, x \in \Theta)$ (see Footnote 3). Picking an arbitrary element $x \in \Theta$ of the frame, a basis of vectors which generate such vector space

$a(\mathcal{P})$ is formed by the $n-1$ vectors: $b_y - b_x$, $\forall y \in \Theta$, $y \neq x$. The non-orthogonality of $a(\bar{b}, \overline{pl}_b)$ and $a(\mathcal{P})$ can therefore be expressed by saying that for at least one of such basis vectors the scalar product $\langle \cdot \rangle$ with the difference vector $\overline{pl}_b - \bar{b}$ (the generator of the line $a(\bar{b}, \overline{pl}_b)$) is non-zero:

$$\exists y \neq x \in \Theta \;\; s.t. \;\; \langle \overline{pl}_b - \bar{b}, b_y - b_x \rangle \neq 0. \tag{25}$$

Recall that $\phi_1[b]$, as defined in (24), is the angle between $a(\bar{b}, \overline{pl}_b)$ and a specific line $a(\widetilde{b}, \widetilde{pl_b})$ laying on the probabilistic subspace.
The condition under which orthogonality holds has a significant interpretation in terms of the uncertainty expressed the belief function b on the probability value of each singleton.

Theorem 3. *$a(\bar{b}, \overline{pl}_b) \perp \mathcal{P}$ (and therefore $\phi_1[b] = \pi/2$) iff*

$$\sum_{A \supsetneq x} m_b(A) = pl_b(x) - m_b(x) = const \quad \forall\, x \in \Theta.$$

If b is Bayesian, $pl_b(x) - m_b(x) = 0 \; \forall x \in \Theta$. If b is *not* Bayesian, there exists at least a singleton x such that $pl_b(x) - m_b(x) > 0$. In this case we can define the probability function

$$R[b] = \sum_{x \in \Theta} \frac{pl_b(x) - m_b(x)}{k_{pl_b} - k_{m_b}} b_x = \frac{\overline{pl}_b - \bar{b}}{k_{pl_b} - k_{m_b}}. \tag{26}$$

$R[b]$ is a probability for $\frac{pl_b(x) - m_b(x)}{k_{pl_b} - k_{m_b}} \geq 0$ for all $x \in \Theta$, while $\sum_x R[b](x) = \sum_x \frac{pl_b(x) - m_b(x)}{k_{pl_b} - k_{m_b}} = \frac{k_{pl_b} - k_{m_b}}{k_{pl_b} - k_{m_b}} = 1$. The value $R[b](x)$ indicates how much the uncertainty $pl_b(x) - m_b(x)$ on the probability value on x "weights" on the total uncertainty on the probabilities of singletons. It is the natural to call it *relative uncertainty on the probabilities of singletons.* When b is Bayesian, $R[b]$ does not exist.

Corollary 1. *The dual line $a(\bar{b}, \overline{pl}_b)$ is orthogonal to $\mathcal{P}$ iff the relative uncertainty on the probabilities of singletons is the uniform probability: $R[b](x) = 1/|\Theta|$ for all $x \in \Theta$.*

In this case the evidence carried by b yields the same uncertainty on the probability value of all singletons. By definition of $p[b]$ (11)

$$\begin{aligned} p[b](x) &= m_b(x) + \beta[b](pl_b(x) - m_b(x)) \\ &= m_b(x) + \frac{1 - k_{m_b}}{\sum_{y \in \Theta}(pl_b(y) - m_b(y))}(pl_b(x) - m_b(x)) \\ &= m_b(x) + (1 - k_{m_b}) R[b](x) = m_b(x) + \frac{1 - k_{m_b}}{n} \end{aligned}$$

the intersection probability re-assigns the (non specific) mass originally given by b to non-singletons to each singleton on equal basis.

Dependence of ϕ_2 on the relative uncertainty
The value of $\phi_2[b]$ also depends on the relative uncertainty on the probabilities of singletons.

Theorem 4. *Denote by* $\mathbf{1} = pl_\Theta$ *the vector* $[1,..,1]'$. *Then*

$$\cos(\pi - \phi_2[b]) = 1 - \frac{\langle \mathbf{1}, R[b] \rangle}{\|R[b]\|^2}, \tag{27}$$

where again $\langle \mathbf{1}, R[b] \rangle$ *denotes the scalar product between the unit vector* $\mathbf{1} = [1,..,1]' \in \mathbb{R}^{N-2}$ *and the vector* $R[b] \in \mathbb{R}^{N-2}$.

We can observe that:
1. $\phi_2[b] = \pi$ ($\cos = 1$) iff $\langle \mathbf{1}, R[b] \rangle = 0$. But this never happens, as $\langle \mathbf{1}, p \rangle = 2^{n-1} - 1$ $\forall p \in \mathcal{P}$ (see proof of Theorem 4).
2. $\phi_2[b] = 0$ ($\cos = -1$) iff $\|R[b]\|^2 = \langle \mathbf{1}, R[b] \rangle / 2$. This condition also never materializes for belief functions defined on non-trivial frames of discernment.

Theorem 5. $\phi_2[b] \neq 0$ *and the lines* $a(\overline{b}, \overline{pl}_b)$, $a(\widehat{b}, \widehat{pl}_b)$ *never coincide for any* $b \in \mathcal{B}$ *when* $|\Theta| > 2$, *while* $\phi_2[b] = 0$ $\forall b \in \mathcal{B}$ *whenever* $|\Theta| \leq 2$.

Let us see that by comparing the situations of the 2-element and 3-element frames. If $\Theta = \{x, y\}$ we have that $pl_b(x) - m_b(x) = m_b(\Theta) = pl_b(y) - m_b(y)$ so that

$$R[b] = \frac{1}{2} b_x + \frac{1}{2} b_y = \overline{\mathcal{P}} \quad \forall b \in \mathcal{B}$$

(where $\overline{\mathcal{P}}$ denotes the uniform probability on Θ, Figure 4) and $R[b] = \frac{1}{2}\mathbf{1} = \frac{1}{2} pl_\Theta$. In the binary case the angle $\phi_2[b]$ is zero for all belief functions. As we observed before, $b = \overline{b} = \widehat{pl}_b$, $pl_b = \overline{pl}_b = \widehat{b}$ and the geometry of the epistemic family is a planar one.
On the other side, if $\Theta = \{x, y, z\}$ not even the vacuous belief function b_Θ (such that $m_{b_\Theta}(\Theta) = 1$) meets condition 2. In that case $R[b_\Theta] = \overline{\mathcal{P}} = \frac{1}{3} b_x + \frac{1}{3} b_y + \frac{1}{3} b_z$ and R is still the uniform probability. But $\langle R[b_\Theta], \mathbf{1} \rangle = 3$, while

$$\langle R[b_\Theta], R[b_\Theta] \rangle = \langle \overline{\mathcal{P}}, \overline{\mathcal{P}} \rangle = \Big\langle \Big[\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3}\Big]', \Big[\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3}\Big]' \Big\rangle = \frac{15}{9}.$$

Unifying condition for the epistemic family and the angle $\phi_3[b]$. The angle $\phi_3[b]$ is instead related to the condition under which relative plausibility of singletons and relative belief of singletons coincide: the analogous of Proposition 1 for the "affine" family of probability

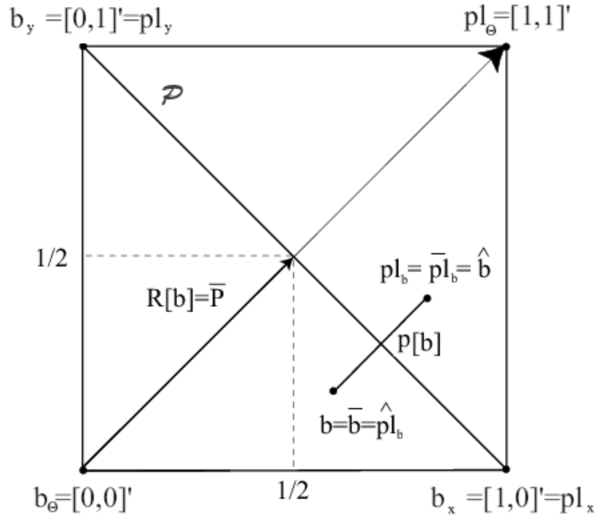

Figure 4. $\phi_2[b] = 0$ for all belief functions in the size-two frame $\Theta = \{x, y\}$, as $R[b] = [1/2, 1/2]'$ is parallel to $pl_\Theta = \mathbf{1}$ for all b.

approximations. As a matter of fact, the angle $\phi_3[b]$ is equal to zero iff $\tilde{b} = \widetilde{pl}_b$, which is equivalent to

$$m_b(x)/k_{m_b} = pl_b(x)/k_{pl_b} \quad \forall x \in \Theta.$$

Again, this necessary and sufficient condition for $\phi_3[b] = 0$ can expressed in terms of $R[b]$, as

$$\begin{array}{rcl} R[b](x) &=& (pl_b(x) - m_b(x))/(k_{pl_b} - k_{m_b}) \\ &=& \frac{1}{k_{pl_b} - k_{m_b}}\Big(\frac{k_{pl_b}}{k_{m_b}} m_b(x) - m_b(x)\Big) = m_b(x)/k_{m_b} \quad \forall x \in \Theta, \end{array} \tag{28}$$

i.e., $R[b] = \tilde{b}$, with $R[b]$ "squashing" $\widetilde{pl}_b$ onto $\tilde{b}$ from the outside. In this case the quantities $\overline{pl}_b, \widehat{pl}_b, \widetilde{pl}_b, p[b], \bar{b}, \hat{b}, \tilde{b}$ all lie in the same plane.

5.5. Singular case

We need to pay some attention to the singular case (from a geometric point of view) in which the relative belief of singletons simply does not exist, $k_{m_b} = \sum_x m_b(x) = 0$. One can argue that the existence of the relative belief of singletons is subject to quite a strong condition: $k_{m_b} \neq 0$. However, it can be proven that the case in which $\tilde{b}$ does not exist is indeed pathological, as it excludes a great deal of belief and probability measures (Cuzzolin, 2008c).

We can note that, even in this case, the belief of singletons $\bar{b}$ still exists, and by Equation (18)

$$\bar{b} = b_\Theta$$

while $\hat{b} = pl_\Theta$ by duality. Recall the description in terms of planes we gave in Section 5.3. In this case the first two planes $a(b, p[b], \hat{b}) = a(a(\hat{b}, \widehat{pl}_b), a(\bar{b}, \overline{pl}_b)) = a(a(b_\Theta, \widehat{pl}_b), a(pl_\Theta, \overline{pl}_b)) = a(b_\Theta, \widetilde{pl}_b, pl_\Theta)$ coincide, while the third one $a(b_\Theta, \tilde{b}, pl_\Theta)$ simply does not exist. The ge-

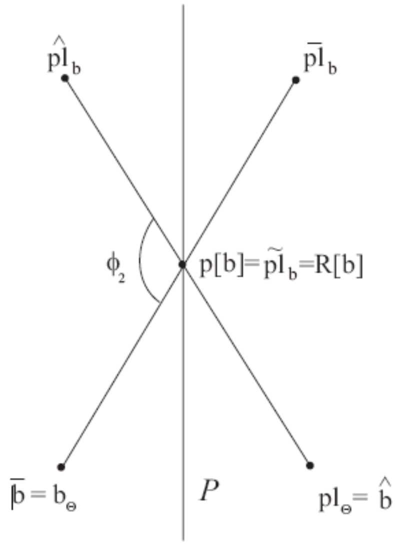

Figure 5. Geometry of relative plausibility of singletons, relative uncertainty on the probabilities of singletons and intersection probability in the singular case when $k_{m_b} = \sum_x m(x) = 0$.

ometry of the epistemic family of transformations in the singular case reduces to a planar geometry (see Figure 5), which depends only on the angle $\phi_2[b]$. It is remarkable that, in this case,

$$p[b](x) = m_b(x) + \frac{1 - k_{m_b}}{k_{pl_b} - k_{m_b}}(pl_b(x) - m_b(x)) = \frac{1}{k_{pl_b}} pl_b(x) = \widetilde{pl}_b(x).$$

Theorem 6. *If a belief function b does not admit relative belief of singletons (as b assigns zero mass to all singletons) then its relative plausibility of singletons and intersection probability coincide.*

Also, in this case the relative uncertainty on the probabilities of singletons coincides with the relative plausibility of singletons too: $R[b] = \widetilde{pl}_b = p[b]$ (see Equation (26)).

6. Geometry in the probability simplex

The geometry of relative belief and relative plausibility of singletons in the space of all (pseudo) belief functions is a function of three angles and planes. As they are probability measures, however, it is obviously interesting to understand their behavior *in the simplex of all probabilities.* We can note for instance that, as

$$R[b](k_{pl_b} - k_{m_b}) = \overline{pl}_b - \overline{b} = \widetilde{pl}_b\, k_{pl_b} - \widetilde{b}\, k_{m_b} =$$
$$= \widetilde{pl}_b\, k_{pl_b} - \widetilde{b}\, k_{m_b} + k_{pl_b}\, \widetilde{b} - k_{pl_b}\, \widetilde{b} = k_{pl_b}\, (\widetilde{pl}_b - \widetilde{b}) + \widetilde{b}(k_{pl_b} - k_{m_b}),$$

$R[b]$ always lies on the line joining $\widetilde{b}$ and $\widetilde{pl}_b$:

$$R[b] = \widetilde{b} + \frac{k_{pl_b}}{k_{pl_b} - k_{m_b}}(\widetilde{pl}_b - \widetilde{b}). \tag{29}$$

Let us study the situation in a simple example.

6.1. Geometry in the 3-element frame

Consider a belief function b_1 with basic belief assignment

$$m_{b_1}(x) = 0.5,\;\; m_{b_1}(y) = 0.1,\;\; m_{b_1}(\{x, y\}) = 0.3,\;\; m_{b_1}(\{y, z\}) = 0.1$$

on $\Theta = \{x, y, z\}$. The probability intervals of the singletons have widths

$$\begin{array}{rcl} pl_{b_1}(x) - m_{b_1}(x) & = & m_{b_1}(\{x, y\}) = 0.3, \\ pl_{b_1}(y) - m_{b_1}(y) & = & m_{b_1}(\{x, y\}) + m_{b_1}(\{y, z\}) = 0.4, \\ pl_{b_1}(z) - m_{b_1}(z) & = & m_{b_1}(\{y, z\}) = 0.1. \end{array}$$

Their relative uncertainty is therefore: $R[b_1](x) = 3/8$, $R[b_1](y) = 1/2$, $R[b_1](z) = 1/8$. $R[b_1]$ is plotted as a point of the probability simplex $\mathcal{P} = Cl(b_x, b_y, b_z)$ in Figure 6. Its L_2 distance from the uniform probability $\overline{\mathcal{P}} = [1/3, 1/3, 1/3]'$ in $\mathcal{P}$ is clearly

$$\begin{array}{rcl} \|\overline{\mathcal{P}} - R[b_1]\| & = & \Big[\sum_x (1/3 - R[b_1](x))^2\Big]^{1/2} \\ & = & \Big[\Big(\frac{1}{3} - \frac{3}{8}\Big)^2 + \Big(\frac{1}{3} - \frac{1}{2}\Big)^2 + \Big(\frac{1}{3} - \frac{1}{8}\Big)^2\Big]^{1/2} = 0.073. \end{array}$$

The related intersection probability (as $k_{m_{b_1}} = 0.6$, $k_{pl_{b_1}} = 0.8 + 0.5 + 0.1 = 1.4$, $\beta[b_1] = (1 - 0.6)/(1.4 - 0.6) = 1/2$) is

$$p[b_1](x) = 0.5 + \tfrac{1}{2}0.3 = 0.65,\quad p[b_1](y) = 0.1 + \tfrac{1}{2}0.4 = 0.3,$$
$$p[b_1](z) = 0 + \tfrac{1}{2}0.1 = 0.05,$$

and is plotted as a square (the second from the left) on the dotted triangle of Figure 6.
A larger uncertainty on the probability of singletons is achieved by b_2

$$m_{b_2}(x) = 0.5, \;\; m_{b_2}(y) = 0.1, \;\; m_{b_2}(z) = 0, \;\; m_{b_2}(\{x, y\}) = 0.4,$$

in which all the higher-size mass is assigned to a single focal element $\{x, y\}$. In that case $pl_{b_2}(x) - m_{b_2}(x) = 0.4$, $pl_{b_2}(y) - m_{b_2}(y) = 0.4$, $pl_{b_2}(z) - m_{b_2}(z) = 0$, so that the relative uncertainty on the probabilities of singletons is $R[b_2](x) = 1/2$, $R[b_2](y) = 1/2$, $R[b_2](z) = 0$ with a distance from $\overline{\mathcal{P}}$ of $[(1/6)^2 + (1/6)^2 + (1/3)^2]^{1/2} = 0.408$.
The corresponding intersection probability (as $\beta[b_2] = (1 - 0.6)/0.8$ is still 1/2) is the first square from the left on the same dotted triangle:

$$p[b_2](x) = 0.5 + \frac{1}{2}0.4 = 0.7, \;\; p[b_2](y) = 0.1 + \frac{1}{2}0.4 = 0.3, \;\; p[b_2](z) = 0.$$

If we spread the mass of non-singletons on to two focal elements

$$m_{b_3}(x) = 0.5, \;\; m_{b_3}(y) = 0.1, \;\; m_{b_3}(\{x, y\}) = 0.2, \;\; m_{b_3}(\{y, z\}) = 0.2$$

we get

$$\begin{array}{c} pl_{b_3}(x) - m_{b_3}(x) = 0.2, \quad pl_{b_3}(y) - m_{b_3}(y) = 0.4, \\ pl_{b_3}(z) - m_{b_3}(z) = 0.2 \end{array}$$

which corresponds to $R[b_3](x) = 1/4$, $R[b_3](y) = 1/2$, $R[b_3](z) = 1/4$, whose distance from $\overline{\mathcal{P}}$ is 0.2041. The intersection probability has values $p[b_3](x) = 0.5 + \frac{1}{2}0.2 = 0.6$, $p[b_3](y) = 0.1 + \frac{1}{2}0.4 = 0.3$, $p[b_3](z) = 0 + \frac{1}{2}0.2 = 0.1$.

Fixing the mass of singletons determines a set of admissible belief functions. In the example, b_1, b_2 and b_3 all belong to such a set:

$$\Big\{ b : m_b(x) = 0.5, m_b(y) = 0.1, m_b(z) = 0, \sum_{|A|>1} m_b(A) = 0.4 \Big\}. \quad (30)$$

The corresponding relative uncertainty on the probability of singletons is constrained to live in the simplex delimited by the dashed lines in Figure 6. Of the three b.f.s we have considered, b_2 corresponds to the maximal imbalance between the masses of size-2 focal elements, as it assigns the whole mass to $\{x, y\}$. As a result, $R[b_2]$ has maximal distance from the uniform probability $\overline{\mathcal{P}}$. The belief function b_3 spreads instead the mass equally between $\{x, y\}$ and $\{y, z\}$. As a result, $R[b_3]$ has minimal distance from $\overline{\mathcal{P}}$.
Accordingly, the intersection probability (17) is constrained to live in the simplex delimited by the dotted lines.

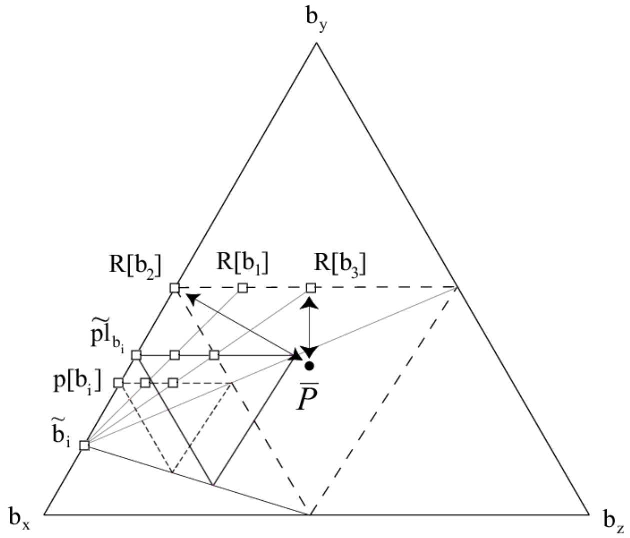

Figure 6. Locations of the members of the epistemic family in the probability simplex $\mathcal{P} = Cl(b_x, b_y, b_z)$ for a 3-element frame $\Theta = \{x, y, z\}$. The relative uncertainty on the probability of singletons $R[b]$, the relative plausibility of singletons $\widetilde{pl}_b$ and the intersection probability $p[b]$ for the family of belief functions on the 3-element frame defined by the mass assignment (30) lie inside the dashed, solid and dotted triangles in the diagram, respectively. The locations of $R[b_1]$, $R[b_2]$, $R[b_3]$ for the three belief functions b_1, b_2 and b_3 discussed in the example are shown. The relative plausibility of singletons and the intersection probability for the same b.f.s appear on the same side of the corresponding triangles in the same order. The relative belief of singletons $\tilde{b}$ lies on the bottom-left square near b_x for all the belief functions of the considered family (30).

All those belief functions have by definition (3) the same relative belief $\tilde{b}$. The lines determined by $R[b]$ and $p[b]$ for each admissible belief function b in the set (30) intersect as a matter of fact in

$$\tilde{b}(x) = 5/6, \quad \tilde{b}(y) = 1/6, \quad b(z) = 0$$

(bottom left square). This is due to the fact that

$$\begin{aligned} p[b] &= \sum_x \Big[m_b(x) + (1 - k_{m_b}) R[b](x) \Big] b_x = \sum_x m_b(x) b_x + (1 - k_{m_b}) R[b] \\ &= k_{m_b} \tilde{b} + (1 - k_{m_b}) R[b] \end{aligned}$$

so that $\tilde{b}$ is collinear with $R[b], p[b]$.
Finally, the associated relative plausibilities of singletons also live in a simplex (solid lines in Figure 6). The probabilities $\widetilde{pl}_{b_1}$, $\widetilde{pl}_{b_2}$, and $\widetilde{pl}_{b_3}$ are plotted as squares located in the same order as above. According to (29), $\widetilde{pl}_b$, $\tilde{b}$, and $R[b]$ are also collinear for all belief functions b.

6.2. Singular case in the 3-element frame

Let us pay some attention to the singular case. For each belief function b such that $m_b(x) = m_b(y) = m_b(z) = 0$ we have

$$\begin{array}{l} pl_b(x) = m_b(\{x,y\}) + m_b(\{x,z\}) + m_b(\Theta) = 1 - m_b(\{y,z\}), \\ pl_b(y) = m_b(\{x,y\}) + m_b(\{y,z\}) + m_b(\Theta) = 1 - m_b(\{x,z\}), \\ pl_b(z) = m_b(\{x,z\}) + m_b(\{y,z\}) + m_b(\Theta) = 1 - m_b(\{x,y\}). \end{array}$$

Hence $pl_b(w) - m_b(w) = pl_b(w)$ for all elements $w \in \Theta$, so that

$$\begin{array}{l} \sum_w (pl_b(w) - m_b(w)) = pl_b(x) + pl_b(y) + pl_b(z) = \\ = 2(m_b(\{x,y\}) + m_b(\{x,z\}) + m_b(\{y,z\})) + 3m_b(\Theta) = 2 + m_b(\Theta) \end{array}$$

and

$$\beta[b] = \frac{1 - \sum_w m_b(w)}{\sum_w (pl_b(w) - m_b(w))} = \frac{1}{\sum_w pl_b(w)} = \frac{1}{2 + m_b(\Theta)}.$$

Therefore

$$\begin{array}{l} R[b](x) = \dfrac{pl_b(x) - m_b(x)}{\sum_w (pl_b(w) - m_b(w))} = \dfrac{1 - m_b(\{y,z\})}{2 + m_b(\Theta)}, \\ \qquad R[b](y) = \dfrac{1 - m_b(\{x,z\})}{2 + m_b(\Theta)}, \quad R[b](z) = \dfrac{1 - m_b(\{x,y\})}{2 + m_b(\Theta)}; \\ p[b](x) = m_b(x) + \beta[b](pl_b(x) - m_b(x)) = \beta[b]pl_b(x) = \dfrac{1 - m_b(\{y,z\})}{2 + m_b(\Theta)}, \\ \qquad p[b](y) = \dfrac{1 - m_b(\{x,z\})}{2 + m_b(\Theta)}, \quad p[b](z) = \dfrac{1 - m_b(\{x,y\})}{2 + m_b(\Theta)}; \\ \widetilde{pl}_b(x) = \dfrac{pl_b(x)}{\sum_w pl_b(w)} = \dfrac{1 - m_b(\{y,z\})}{2 + m_b(\Theta)}, \\ \qquad \widetilde{pl}_b(y) = \dfrac{1 - m_b(\{x,z\})}{2 + m_b(\Theta)}, \ \widetilde{pl}_b(z) = \dfrac{1 - m_b(\{x,y\})}{2 + m_b(\Theta)} \end{array}$$

and $R[b] = \widetilde{pl}_b = p[b]$ as stated by Theorem 6.

We have seen above that in the non-singular case all those quantities live in different simplices that "converge" to $\tilde{b}$. When $\tilde{b}$ does not exist, all such simplices coincide. In the ternary case, for a given value of $m_b(\Theta)$, this is the triangle with vertices

$$\begin{array}{c} \left[\frac{1}{2+m_b(\Theta)}, \frac{1}{2+m_b(\Theta)}, \frac{m_b(\Theta)}{2+m_b(\Theta)}\right]', \left[\frac{1}{2+m_b(\Theta)}, \frac{m_b(\Theta)}{2+m_b(\Theta)}, \frac{1}{2+m_b(\Theta)}\right]', \\ \left[\frac{m_b(\Theta)}{2+m_b(\Theta)}, \frac{1}{2+m_b(\Theta)}, \frac{1}{2+m_b(\Theta)}\right]'. \end{array} \qquad (31)$$

As a reference, for $m_b(\Theta) = 0$ the latter is the triangle delimited by the points p_1, p_2, p_3 in Figure 7 (solid line). For $m_b(\Theta) = 1$ we get a single

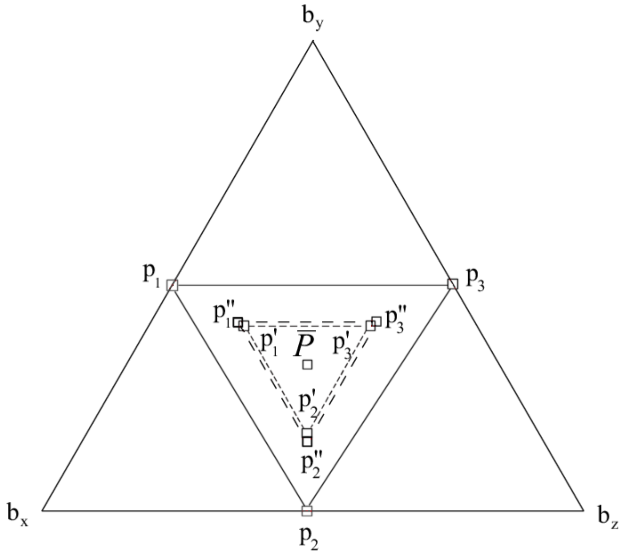

Figure 7. Simplices spanned by $R[b] = p[b] = \widetilde{pl}_b$ and $BetP[b] = \pi[b]$ in the probability simplex for the cardinality 3 frame in the singular case $m_b(x) = m_b(y) = m_b(z) = 0$, for different values of $m_b(\Theta)$. In solid line: The triangle spanned by $R[b] = p[b] = \widetilde{pl}_b$ coincides with that spanned by $BetP[b] = \pi[b]$ for all b such that $m_b(\Theta) = 0$. For $m_b(\Theta) = 1/2$, $R[b] = p[b] = \widetilde{pl}_b$ spans the triangle $Cl(p'_1, p'_2, p'_3)$ (dotted lines) while $BetP[b] = \pi[b]$ spans the triangle $Cl(p''_1, p''_2, p''_3)$ (dashed lines). For all b s.t. $m_b(\Theta) = 1$ both groups of transformations reduce to the single point $\overline{\mathcal{P}}$.

point: $\overline{\mathcal{P}}$ (the central black square in the Figure). For $m_b(\Theta) = 1/2$, instead, (31) yields

$$Cl(p'_1, p'_2, p'_3) = Cl([2/5, 2/5, 1/5]', [2/5, 1/5, 2/5]', [1/5, 2/5, 2/5]')$$

(dotted triangle). For comparison, let us compute the values of Smets' pignistic probability (13) (which in the 3-element case coincides with the orthogonal projection (Cuzzolin, 2007)). We have

$$\begin{array}{rcl} BetP[b](x) & = & \frac{m_b(\{x,y\})+m_b(\{x,z\})}{2} + \frac{m_b(\Theta)}{3}, \\ BetP[b](y) & = & \frac{m_b(\{x,y\})+m_b(\{y,z\})}{2} + \frac{m_b(\Theta)}{3}, \\ BetP[b](z) & = & \frac{m_b(\{x,z\})+m_b(\{y,z\})}{2} + \frac{m_b(\Theta)}{3}, \end{array}$$

so that the simplices spanned by the pignistic function for the same sample values of $m_b(\Theta)$ are (Figure 7 again): $m_b(\Theta) = 1 \rightarrow \overline{\mathcal{P}}$; $m_b(\Theta) = 0 \rightarrow Cl(p_1, p_2, p_3)$; $m_b(\Theta) = 1/2 \rightarrow Cl(p''_1, p''_2, p''_3)$ where

$$p''_1 = [5/12, 5/12, 1/6]', p''_2 = [5/12, 1/6, 5/12]', p''_3 = [1/6, 5/12, 5/12]'$$

(the vertices of the dashed triangle in the figure). The behavior of the two families of probability transformations is rather similar, at least in the singular case. In both cases approximations are allowed to span only a proper subset of the probability simplex $\mathcal{P}$, stressing the pathological situation of the singular case itself.

7. Equality conditions for both families of approximations

The rich tapestry of results of Sections 5 and 6 complements our knowledge of the geometry of the relation between belief functions and their probability transformations which started with the affine family (Cuzzolin, 2007).
The epistemic family is formed by transformations which depend on the balance between the total plausibility k_{pl_b} of the elements of the frame, and the total mass k_{m_b} assigned to them. This measure of the relative uncertainty on the probabilities of singletons is symbolized by the probability distribution $R[b]$. The examples of Section 6 shed some light on the relative behavior of all probability transformations, at least in the probability simplex. In Section 4 we stressed the different semantics of the two groups. It is now worth to understand under which conditions the probabilities generated by transformations of different families reduce to the same probability distribution. Theorem 6 is a first step in this direction: when b does not admit relative belief, its relative plausibility $\widetilde{pl}_b$ and intersection probability $p[b]$ coincide. We draw again inspiration from the binary case.

7.1. Equal plausibility distribution in the affine family

Let us first focus on functions of the affine family. In particular, let us consider the orthogonal projection of b onto $\mathcal{P}$ (Cuzzolin, 2007)

$$\pi[b](x) = \sum_{A \supseteq \{x\}} m_b(A) \Big(\frac{1 + |A^c| 2^{1-|A|}}{n} \Big) + \sum_{A \not\supseteq \{x\}} m_b(A) \Big(\frac{1 - |A| 2^{1-|A|}}{n} \Big) \tag{32}$$

and the pignistic transformation (13). We can prove that

Lemma 1. *The difference $\pi[b](x) - BetP[b](x)$ between the probability values of orthogonal projection and pignistic function is*

$$\sum_{A \subseteq \Theta} m_b(A) \Big(\frac{1 - |A| 2^{1-|A|}}{n} \Big) - \sum_{A \supseteq \{x\}} m_b(A) \Big(\frac{1 - |A| 2^{1-|A|}}{|A|} \Big). \tag{33}$$

An immediate consequence of (33) is that

Theorem 7. *Orthogonal projection and pignistic function coincide iff*

$$\sum_{A\supseteq\{x\}} m_b(A)(1-|A|2^{1-|A|})\frac{|A^c|}{|A|} = \sum_{A\not\supset\{x\}} m_b(A)(1-|A|2^{1-|A|}) \ \ \forall x\in\Theta. \tag{34}$$

Proof. By Equation (33) the condition $\pi[b](x) - BetP[b](x) = 0$ for all $x \in \Theta$ reads as

$$\sum_{A\subseteq\Theta} m_b(A)\Big(\frac{1-|A|2^{1-|A|}}{n}\Big) = \sum_{A\supseteq\{x\}} m_b(A)\Big(\frac{1-|A|2^{1-|A|}}{|A|}\Big) \quad \forall x\in\Theta$$

i.e.,

$$\begin{array}{rcl}\displaystyle\sum_{A\not\supset\{x\}} m_b(A)\Big(\frac{1-|A|2^{1-|A|}}{n}\Big) & = & \displaystyle\sum_{A\supseteq\{x\}} m_b(A)(1-|A|2^{1-|A|})\Big(\frac{1}{|A|}-\frac{1}{n}\Big) \\ \displaystyle\sum_{A\not\supset\{x\}} m_b(A)\Big(\frac{1-|A|2^{1-|A|}}{n}\Big) & = & \displaystyle\sum_{A\supseteq\{x\}} m_b(A)(1-|A|2^{1-|A|})\Big(\frac{n-|A|}{|A|n}\Big)\end{array}$$

for all singletons $x \in \Theta$, i.e., (34). □

Theorem 7 gives an exhaustive but rather arid description of the relation between $\pi[b]$ and $BetP[b]$. More significant sufficient conditions can be given in terms of belief values. Let us denote by

$$pl_b(x;k) = \sum_{A\supset\{x\},|A|=k} m_b(A)$$

the support focal elements of size k provide to each singleton x.

Corollary 2. *Each of the following is a sufficient condition for the equality of the pignistic and orthogonal transformations of a belief function b ($BetP[b] = \pi[b]$):*

1. *$m_b(A) = 0$ for all $A \subseteq \Theta$ such that $|A| \neq 1, 2, n$;*
2. *the mass of b is equally distributed among all the focal elements $A \subseteq \Theta$ of the same size $|A| = k$, for all sizes $k = 3, ..., n-1$:*

$$m_b(A) = \frac{\sum_{|B|=k} m_b(B)}{\binom{n}{k}}, \ \ \forall A : |A| = k, \ \ \forall k = 3, .., n-1;$$

3. *for all singletons $x \in \Theta$, and for all $k = 3, .., n-1$*

$$pl_b(x;k) = const = pl_b(\cdot;k). \tag{35}$$

If mass is equally distributed among higher-size events the orthogonal projection *is* the pignistic function (Condition 2). The probability closest to b (in the Euclidean sense) is also the barycenter of the simplex $\mathcal{P}[b]$ of consistent probabilities.
This is also the case when events of the same size contribute with the same amount to the plausibility of each singleton (Condition 3).
It is easy to see that Condition 1 implies (is stronger than) Condition 2 which in turn implies Condition 3. All of them are met by belief functions on size-2 frames. In particular, Corollary 2 implies that

Corollary 3. *$BetP[b] = \pi[b]$ for $|\Theta| \leq 3$.*

7.2. Equal plausibility distribution as a general condition

Equal distribution of plausibility (Equation (35)) gives in fact an equality condition for probability transformations of b of *both families*. Consider again the binary case of Figure 1. We can appreciate that belief functions with $m_b(x) = m_b(y)$ lay on the bisector of the first quadrant, which is orthogonal to $\mathcal{P}$. Their relative plausibility is then equal to their orthogonal projection $\pi[b]$.

Theorem 3 can indeed be interpreted in terms of equal distribution of plausibility among singletons. If Equation (35) is met for all $k = 2, ..., n-1$ (this is trivially true for $k = n$) then the uncertainty $pl_b(x) - m_b(x) = \sum_{A \supsetneq \{x\}} m_b(A)$ on the probability value of each singleton $x \in \Theta$ becomes

$$\sum_{A \supsetneq \{x\}} m_b(A) = \sum_{k=2}^{n} \sum_{|A|=k, A \supset \{x\}} m_b(A) = m_b(\Theta) + \sum_{k=2}^{n-1} pl_b(\cdot; k), \quad (36)$$

which is constant for all $x \in \Theta$.
The following is then a consequence of Theorem 3 and Equation (36).

Corollary 4. *If $pl_b(x;k) = const$ for all $x \in \Theta$ and for all $k = 2, ..., n-1$ then the dual line $a(\overline{b}, \overline{pl}_b)$ is orthogonal to $\mathcal{P}$, and the relative uncertainty on the probabilities of the singletons is the uniform probability $R[b] = \overline{\mathcal{P}}$.*

The quantity $pl_b(x;k)$ seems then to be connected to geometric orthogonality in the belief space.
We say that a belief function $b \in \mathcal{B}$ is orthogonal to $\mathcal{P}$ when the vector $\overrightarrow{b\,\mathbf{0}}$ joining the origin $\mathbf{0}$ of $\mathbb{R}^{N-2}$ with b is orthogonal to it.
In (Cuzzolin, 2007) we showed that this is the case if and only if

$$\sum_{A \supset \{y\}, A \not\supset \{x\}} m_b(A) 2^{1-|A|} = \sum_{A \supset \{x\}, A \not\supset \{y\}} m_b(A) 2^{1-|A|} \quad (37)$$

for each pair of distinct singletons $x, y \in \Theta$, $x \neq y$. For instance, the uniform Bayesian belief function $\overline{\mathcal{P}}$ is orthogonal to $\mathcal{P}$.
Again a sufficient condition for (37) can be given in terms of equal distribution of plausibility. Confirming the intuition given by the binary case, in this case all probability transformations of b *converge to the same probability.*

Theorem 8. *If $pl_b(x;k) = const = pl_b(\cdot;k)$ for all $k = 1, ..., n-1$ then b is orthogonal to $\mathcal{P}$, and*

$$\widetilde{pl}_b = R[b] = \pi[b] = BetP[b] = \overline{\mathcal{P}}. \tag{38}$$

In conclusion, if focal elements of the same size equally contribute to the plausibility of each singleton ($pl_b(x;k) = const$) for a certain range of values of $|A| = k$ the following consequences on the relation between all probability transformations and their geometry hold:

$$\forall k = 3, ..., n \ : \ BetP[b] = \pi[b];$$

$$\forall k = 2, ..., n \ : \ a(\overline{b}, \overline{pl}_b) \bot \mathcal{P};$$

$$\forall k = 1, ..., n \ : \ b \bot \mathcal{P}, \widetilde{pl}_b = \widetilde{b} = R[b] = \overline{\mathcal{P}} = BetP[b] = p[b] = \pi[b].$$

Less binding conditions will may be harder to formulate, but are worth to be studied in the near future.

8. Conclusions

Each belief function is associated with two different families of probability transformations, marked by the operator they commute with: affine combination or Dempster's rule. The affine family $p[b]$, $\pi[b]$, $BetP[b]$ is inherently related to the relative locations of belief and plausibility functions. The epistemic family $\widetilde{b}$, $\widetilde{pl}_b$, $R[b]$ has a geometry that can be described in terms of three planes and angles which depend on the relative uncertainty on the probability of the singletons, measured by $R[b]$. Unifying conditions for all transformations of both families can be given by means of the notion of equal plausibility distribution.
The results provided here, nevertheless, give only pointwise information about the difference between distinct approximations. In (Cuzzolin, 2007) we started working on a quantitative analysis of these differences as functions of the basic belief assignment of the original belief function. A complete, exhaustive quantitative comparison of all probability transformations is the natural arrival point of this line of research.

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Appendix

Proof of Theorem 2

By Equation (20) $\widehat{b} - \overline{b} = (1 - k_{m_b})pl_\Theta$ and $\widehat{pl}_b - \overline{pl}_b = (1 - k_{pl_b})pl_\Theta$. Hence $\beta[b](\widehat{pl}_b - \widehat{b}) + \widehat{b}$ is equal to

$$\begin{array}{l}\beta[b]\Big[\overline{pl}_b + (1 - k_{pl_b})pl_\Theta - \overline{b} - (1 - k_{m_b})pl_\Theta\Big] + \overline{b} + (1 - k_{m_b})pl_\Theta = \\ \quad = \beta[b]\Big[\overline{pl}_b - \overline{b} + (k_{m_b} - k_{pl_b})pl_\Theta\Big] + \overline{b} + (1 - k_{m_b})pl_\Theta = \\ \quad = \overline{b} + \beta[b](\overline{pl}_b - \overline{b}) + pl_\Theta\Big[\beta[b](k_{m_b} - k_{pl_b}) + 1 - k_{m_b}\Big].\end{array}$$

But by definition of $\beta[b]$ (12)

$$\beta[b](k_{m_b} - k_{pl_b}) + 1 - k_{m_b} = \frac{1 - k_{m_b}}{k_{pl_b} - k_{m_b}}(k_{m_b} - k_{pl_b}) + 1 - k_{m_b} = 0$$

and (21) is met.

Proof of Theorem 3

By definition of b_A ($b_A(C) = 1$ if $C \supseteq A$, 0 otherwise) we have that

$$\langle b_A, b_B \rangle = \sum_{C \subseteq \Theta} b_A(C) b_B(C) = \sum_{C \supseteq A,B} 1 \cdot 1 = \sum_{C \supseteq A \cup B} 1 = \|b_{A\cup B}\|^2.$$

The scalar product of interest can then be written $\forall y \neq x$ as

$$\begin{array}{rcl}\langle \overline{pl}_b - \overline{b}, b_y - b_x \rangle &=& \Big\langle \sum_{z\in\Theta} (pl_b(z) - m_b(z))\, b_z, b_y - b_x \Big\rangle \\ &=& \sum_{z\in\Theta} (pl_b(z) - m_b(z))\, [\langle b_z, b_y\rangle - \langle b_z, b_x\rangle] \\ &=& \sum_{z\in\Theta} (pl_b(z) - m_b(z))\, \Big[\|b_{z\cup y}\|^2 - \|b_{z\cup x}\|^2\Big].\end{array}$$

We can distinguish three cases:

- if $z \neq x, y$ then $|z \cup x| = |z \cup y| = 2$ and the difference $\|b_{z\cup x}\|^2 - \|b_{z\cup y}\|^2$ goes to zero;

– if $z = x$ then $\|b_{z\cup x}\|^2 - \|b_{z\cup y}\|^2 = \|b_x\|^2 - \|b_{x\cup y}\|^2 = (2^{n-2}-1) - (2^{n-1}-1) = -2^{n-2}$ where $n = |\Theta|$;

– if instead $z = y$ then $\|b_{z\cup x}\|^2 - \|b_{z\cup y}\|^2 = \|b_{x\cup y}\|^2 - \|b_y\|^2 = 2^{n-2}$.

Hence for all $y \neq x$

$$\langle \overline{pl}_b - \bar{b}, b_y - b_x\rangle = 2^{n-2}(pl_b(y) - m_b(y)) - 2^{n-2}(pl_b(x) - m_b(x))$$

and as $\sum_{A\supsetneq x} m_b(A) = pl_b(x) - m_b(x)$ the thesis follows.

PROOF OF COROLLARY 1

As a matter of fact $\sum_{x\in\Theta}\sum_{A\supset x, A\neq x} m_b(A) = \sum_{x\in\Theta}(pl_b(x) - m_b(x)) = k_{pl_b} - k_{m_b}$ so that the condition of Theorem 3 can be written as

$$pl_b(x) - m_b(x) = \sum_{A\supset x, A\neq x} m_b(A) = \frac{k_{pl_b} - k_{m_b}}{n} \;\; \forall x.$$

Replacing this in (26) yields $R[b] = \sum_{x\in\Theta}\frac{1}{n} b_x$.

PROOF OF THEOREM 4

By Equation (21) $p[b] = \bar{b} + \beta[b](\overline{pl}_b - \bar{b})$. After recalling that $\beta[b] = (1 - k_{m_b})(k_{pl_b} - k_{m_b})$ we can write

$$\begin{array}{rcl} \overline{pl}_b - p[b] & = & \overline{pl}_b - \Big[\bar{b} + \beta[b](\overline{pl}_b - \bar{b})\Big] = (1 - \beta[b])(\overline{pl}_b - \bar{b}) \\ & = & \dfrac{k_{pl_b} - 1}{k_{pl_b} - k_{m_b}}(\overline{pl}_b - \bar{b}) = (k_{pl_b} - 1)\; R[b] \end{array} \tag{39}$$

by definition (26) of $R[b]$. On the other side, as $\widehat{pl}_b = \overline{pl}_b + (1 - k_{pl_b}) pl_\Theta$ by Equation (20), we get

$$\begin{array}{rcl} \widehat{pl}_b - p[b] & = & (\widehat{pl}_b - \overline{pl}_b) + (\overline{pl}_b - p[b]) \\ & = & \Big[\overline{pl}_b + (1 - k_{pl_b})pl_\Theta - \overline{pl}_b\Big] + (k_{pl_b} - 1)R[b] \\ & = & (1 - k_{pl_b})pl_\Theta + (k_{pl_b} - 1)R[b] = (k_{pl_b} - 1)(R[b] - pl_\Theta). \end{array} \tag{40}$$

Combining (40) and (39) then yields

$$\begin{array}{rcl} \langle \widehat{pl}_b - p[b], \overline{pl}_b - p[b]\rangle & = & \Big\langle (k_{pl_b} - 1)(R[b] - pl_\Theta), (k_{pl_b} - 1)R[b]\Big\rangle \\ & = & (k_{pl_b} - 1)^2\langle R[b] - pl_\Theta, R[b]\rangle \\ & = & (k_{pl_b} - 1)^2\Big(\langle R[b], R[b]\rangle - \langle pl_\Theta, R[b]\rangle\Big) \\ & = & (k_{pl_b} - 1)^2\Big(\langle R[b], R[b]\rangle - \langle \mathbf{1}, R[b]\rangle\Big). \end{array}$$

But now

$$\cos(\pi - \phi_2) = \frac{\langle \widehat{pl}_b - p[b], \overline{pl}_b - p[b] \rangle}{\| \widehat{pl}_b - p[b] \| \| \overline{pl}_b - p[b] \|}$$

where

$$\begin{aligned} \| \widehat{pl}_b - p[b] \| &= \Big[\langle \widehat{pl}_b - p[b], \widehat{pl}_b - p[b] \rangle \Big]^{1/2} \\ &= (k_{pl_b} - 1) \Big[\langle R[b] - pl_\Theta, R[b] - pl_\Theta \rangle \Big]^{1/2} \\ &= (k_{pl_b} - 1) \Big[\langle R[b], R[b] \rangle + \langle pl_\Theta, pl_\Theta \rangle - 2 \langle R[b], pl_\Theta \rangle \Big]^{1/2} \end{aligned}$$

and $\| \overline{pl}_b - p[b] \| = (k_{pl_b} - 1) \| R[b] \|$ by Equation (39). Hence

$$\cos(\pi - \phi_2[b]) = \frac{(\|R[b]\|^2 - \langle \mathbf{1}, R[b] \rangle)}{\|R[b]\| \sqrt{\|R[b]\|^2 + \langle \mathbf{1}, \mathbf{1} \rangle - 2 \langle R[b], \mathbf{1} \rangle}}. \tag{41}$$

We can further simplify this expression by noticing that for all probabilities $p \in \mathcal{P}$ we have $\langle \mathbf{1}, p \rangle = 2^{|\{x\}^c|} - 1 = 2^{n-1} - 1$ while $\langle \mathbf{1}, \mathbf{1} \rangle = 2^n - 2$, so that $\langle \mathbf{1}, \mathbf{1} \rangle - 2 \langle p, \mathbf{1} \rangle = 0$ and being $R[b]$ a probability we get (27).

Proof of Theorem 5

It is more convenient to make use of (41). As $\phi_2[b] = 0$ iff $\cos(\pi - \phi_2[b]) = -1$ the desired condition is

$$-1 = \frac{(\|R[b]\|^2 - \langle \mathbf{1}, R[b] \rangle)}{\|R[b]\| \sqrt{\|R[b]\|^2 + \langle \mathbf{1}, \mathbf{1} \rangle - 2 \langle R[b], \mathbf{1} \rangle}}$$

i.e., after elevating to the square both numerator and denominator, $\|R[b]\|^2 (\|R[b]\|^2 + \langle \mathbf{1}, \mathbf{1} \rangle - 2 \langle R[b], \mathbf{1} \rangle) = \|R[b]\|^4 + \langle \mathbf{1}, R[b] \rangle^2 - 2 \langle \mathbf{1}, R[b] \rangle \|R[b]\|^2$. After erasing the common terms we get that $\phi_2[b]$ is nil if and only if

$$\langle \mathbf{1}, R[b] \rangle^2 = \|R[b]\|^2 \langle \mathbf{1}, \mathbf{1} \rangle. \tag{42}$$

Condition (42) has the form

$$\langle A, B \rangle^2 = \|A\|^2 \|B\|^2 \cos^2(\widehat{AB}) = \|A\|^2 \|B\|^2$$

i.e., $\cos^2(\widehat{AB}) = 1$, with $A = pl_\Theta$, $B = R[b]$. This yields $\cos(\widehat{R[b] pl_\Theta}) = 1$ or $\cos(\widehat{R[b] pl_\Theta}) = -1$, i.e., $\phi_2[b] = 0$ if and only if $R[b]$ is (anti-)parallel to $pl_\Theta = \mathbf{1}$. But this means $R[b] = \alpha \, pl_\Theta$ for some scalar α, i.e.,

$$R[b] = -\alpha \sum_{A \subseteq \Theta} (-1)^{|A|} b_A$$

(since $pl_\Theta = -\sum_{A \subseteq \Theta} (-1)^{|A|} b_A$ by Equation (9)). But $R[b]$ is a probability (i.e., a linear combination of categorical probabilities b_x only) and since the vectors $\{b_A, A \subsetneq \Theta\}$ which represent all categorical belief functions are linearly independent the two conditions are never met together, unless $|\Theta| = 2$.

PROOF OF LEMMA 1

Using the form (32) of the orthogonal projection we get $\pi[b](x) - BetP[b](x) =$

$$= \sum_{A \supseteq \{x\}} m_b(A) \Big(\frac{1 + |A^c| 2^{1-|A|}}{n} - \frac{1}{|A|} \Big) + \sum_{A \not\supset \{x\}} m_b(A) \Big(\frac{1 - |A| 2^{1-|A|}}{n} \Big)$$

but

$$\begin{array}{l} \dfrac{1 + |A^c| 2^{1-|A|}}{n} - \dfrac{1}{|A|} = \dfrac{|A| + |A|(n - |A|) 2^{1-|A|} - n}{n|A|} = \\ = \dfrac{(|A| - n)(1 - |A| 2^{1-|A|})}{n|A|} = \Big(\dfrac{1}{n} - \dfrac{1}{|A|} \Big) \Big(1 - |A| 2^{1-|A|} \Big) \end{array}$$

so that $\pi[b](x) - BetP[b](x) =$

$$= \sum_{A \supseteq \{x\}} m_b(A) \Big(\frac{1 - |A| 2^{1-|A|}}{n} \Big) \Big(1 - \frac{n}{|A|} \Big) + \sum_{A \not\supset \{x\}} m_b(A) \Big(\frac{1 - |A| 2^{1-|A|}}{n} \Big) \tag{43}$$

or equivalently, Equation (33).

PROOF OF COROLLARY 2

Let us consider all claims. Equation (34) can be expanded as follows

$$\begin{array}{l} \displaystyle\sum_{k=3}^{n-1} \Big(1 - k 2^{1-k} \Big) \frac{n-k}{k} \sum_{A \supset \{x\}, |A| = k} m_b(A) = \sum_{k=3}^{n-1} \Big(1 - k 2^{1-k} \Big) \sum_{A \not\supset \{x\}, |A| = k} m_b(A) \\ \displaystyle\quad \equiv \sum_{k=3}^{n-1} \Big(1 - k 2^{1-k} \Big) \Big[\frac{n-k}{k} \sum_{A \supset \{x\}, |A| = k} m_b(A) - \sum_{A \not\supset \{x\}, |A| = k} m_b(A) \Big] = 0 \\ \displaystyle\qquad \equiv \sum_{k=3}^{n-1} \Big(\frac{1 - k 2^{1-k}}{k} \Big) \Big[n \sum_{A \supset \{x\}, |A| = k} - k \sum_{|A| = k} m_b(A) \Big] = 0 \end{array}$$

after noticing that $1 - k \cdot 2^{1-k} = 0$ for $k = 1, 2$ and the coefficient of $m_b(\Theta)$ in Equation (34) is zero, since $|\Theta^c| = |\emptyset| = 0$.
The condition of Theorem 7 can then be rewritten as

$$n \sum_{k=3}^{n-1} \Big(\frac{1 - k 2^{1-k}}{k} \Big) \sum_{A \supset \{x\}, |A| = k} m_b(A) = \sum_{k=3}^{n-1} (1 - k 2^{1-k}) \sum_{|A| = k} m_b(A) \tag{44}$$

for all $x \in \Theta$. Condition 1 follows immediately from (44).
In the case of Condition 2 the equation becomes

$$\sum_{k=3}^{n-1} \frac{n}{k}(1-k2^{1-k})\binom{n-1}{k-1} \frac{\sum_{|A|=k} m_b(A)}{\binom{n}{k}} = \sum_{k=3}^{n-1}(1-k2^{1-k}) \sum_{|A|=k} m_b(A)$$

which is true since $\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$.
Finally let us consider Condition 3. Under (35) the system of equations (44) reduces to a single equation

$$\sum_{k=3}^{n-1}(1-k2^{1-k})\ \frac{n}{k}\ pl_b(.;k) = \sum_{k=3}^{n-1}(1-k2^{1-k}) \sum_{|A|=k} m_b(A).$$

The latter is verified if $\sum_{|A|=k} m_b(A) = \frac{n}{k} pl_b(.;k)\ \forall k = 3, ..., n-1$, which is in turn equivalent to $n\ pl_b(.;k) = k \sum_{|A|=k} m_b(A)\ \forall k = 3, ..., n-1$. Under the hypothesis of the Theorem we get that

$$n\ pl_b(.;k) = k \sum_{|A|=k} m_b(A) = \sum_{|A|=k} m_b(A)|A| = \sum_{x\in\Theta}\ \sum_{A\supset\{x\},|A|=k} m_b(A).$$

Proof of Theorem 8

Condition (37) is equivalent to

$$\sum_{A\supseteq\{y\}} m_b(A)2^{1-|A|} = \sum_{A\supseteq\{x\}} m_b(A)2^{1-|A|} \equiv \sum_{k=1}^{n-1} \frac{1}{2^k} \sum_{|A|=k,A\supset\{y\}} m_b(A) =$$
$$= \sum_{k=1}^{n-1} \frac{1}{2^k} \sum_{|A|=k,A\supset\{x\}} m_b(A) \equiv \sum_{k=1}^{n-1} \frac{1}{2^k}\ pl_b(y;k) = \sum_{k=1}^{n-1} \frac{1}{2^k}\ pl_b(x;k)$$

for all $y \neq x$, and if $pl_b(x;k) = pl_b(y;k)\ \forall y \neq x$ the condition is met.

To prove the equality (38) let us rewrite the values of the pignistic function $BetP[b](x)$ in terms of $pl_b(x;k)$ as

$$BetP[b](x) = \sum_{A\supseteq\{x\}} \frac{m_b(A)}{|A|} = \sum_{k=1}^{n} \sum_{A\supset\{x\},|A|=k} \frac{m_b(A)}{k} = \sum_{k=1}^{n} \frac{pl_b(x;k)}{k}$$

which is constant under the hypothesis, yielding $BetP[b] = \overline{\mathcal{P}}$. Also, as

$$pl_b(x) = \sum_{A\supseteq\{x\}} m_b(A) = \sum_{k=1}^{n} \sum_{A\supset\{x\},|A|=k} m_b(A) = \sum_{k=1}^{n} pl_b(x;k)$$

we get

$$\widetilde{pl}_b(x) = \frac{pl_b(x)}{k_{pl_b}} = \frac{\sum_{k=1}^n pl_b(x;k)}{\sum_{x\in\Theta}\sum_{k=1}^n pl_b(x;k)}$$

which is equal to $1/n$ if $pl_b(x;k) = pl_b(\cdot;k)\ \forall k, x$.
Finally, under the same condition,

$$\begin{array}{rcl} p[b](x) & = & m_b(x) + \beta[b]\Big(pl_b(x) - m_b(x)\Big) \\ & = & pl_b(\cdot;1) + \beta[b]\displaystyle\sum_{k=2}^n pl_b(\cdot;k) = const = \frac{1}{n}; \\ \tilde{b}(x) & = & \dfrac{m_b(x)}{k_{m_b}} = \dfrac{pl_b(x;1)}{\displaystyle\sum_{y\in\Theta} pl_b(y;1)} = \dfrac{pl_b(\cdot;1)}{npl_b(\cdot;1)} = \dfrac{1}{n}. \end{array}$$

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