An algebraic study of the notion of independence of frames

Fabio Cuzzolin
Department of Computing, Oxford Brookes University, Oxford, UK
(`fabio.cuzzolin@brookes.ac.uk`*).*

July 24th, 2009

Abstract. In this paper we discuss the notion of independence of frames in the theory of evidence from an algebraic point of view, in the perspective of employing algebraic techniques for the treatment of conflicting evidence. Families of frames can be given several algebraic interpretations in terms of upper and lower semi-modular lattices. Each of those structures are endowed with a particular extension of classical matroidal independence, which we prove to be distinct even though related to independence of frames. We show that the latter is in fact incompatible with matroidal independence itself.

Keywords: Theory of evidence, belief functions, independence of frames, Dempster's rule, matroid, semi-modular lattice.

1. Introduction

The theory of evidence was born as a contribution to a mathematically rigorous description of subjective probability. In subjective probability, different observers (or "experts") of the same phenomenon possess in general different notions of what the decision space is. Mathematically, this translates into admitting the existence of several distinct representations of this decision space at different levels of refinement. Evidence will in general be available on several of those domains or *frames*. In order for those experts to reach a consensus on the answer to the considered problem, it is necessary for such frames to be mathematically related to each other. This idea is embodied in the theory of evidence by the notion of *family of frames*. The evidence gathered on distinct frames of the family (corresponding to different experts or sensors) can then be moved to a common frame in order to be merged.
In this context the notion of *independence of frames* $\mathcal{IF}$ (Shafer, 1976) plays an important role. If different pieces of evidence are encoded as different belief functions on distinct frames, evidence fusion under Dempster's orthogonal sum (Dempster, 1967; Dempster, 1968; Dempster, 1969) is guaranteed to take place in all cases if and only if the involved frames are independent (Cuzzolin, 2005a) in a very precise way, which derives from Boolean theory. As Dempster's sum assumes

the conditional independence of the underlying probabilities generating belief functions through multi-valued mappings (Dempster, 1967; Dempster, 1968; Dempster, 1969), it is not surprising to realize that combinability (in Dempster's approach) and independence of frames (in Shafer's formulation of the theory of evidence) are strictly intertwined.

The formal definition of evidence combination has been widely studied (Zadeh, 1986; Yager, 1987) in different mathematical frameworks (Smets, 1990; Dubois and Prade, 1992). An exhaustive review would be impossible here. In particular, some work has indeed been done on the issue of merging conflicting evidence (Deutsch-McLeish, 1990; Josang et al., 2003; Lefevre et al., 2002; Wierman, 2001), specially in critical situations in which the latter is derived from dependent sources (Cattaneo, 2003). Campos and de Souza (Campos and de Souza, 2005) presented a method for fusing highly conflicting evidence which overcomes well known counterintuitive results. Liu (Liu, 2006) has formally defined when two basic belief assignments are in conflict by means of quantitative measures of both the mass of the combined belief assigned to the emptyset before normalization, and the distance between betting commitments of beliefs. Murphy (Murphy, 2000), on her side, has studied a related problem: the failure to balance multiple evidence. The notion of conflicting evidence has been widely used in the context of sensor fusion (Carlson and Murphy, 2005): The matter has also been recently surveyed by Sentz and Ferson (Sentz and Ferson, 2002).
On the other side, though, not much work has been done on the properties of the families of compatible frames and the link with evidence combination. In (Shafer et al., 1987) an analysis of the collections of partitions of a given frame in the context of the hierarchical representation of belief can nevertheless be found, while in (Kohlas and Monney, 1995) both the lattice-theoretical interpretation of families of frames and the meaning of the concept of independence have been discussed. In (Cuzzolin, 2005a) these themes were reconsidered: the structure of Birkhoff lattice of a family of frames was proven, and the crucial relation between Dempster's combination and independence of frames highlighted.

1.1. Contribution

Here we build on the results obtained in (Cuzzolin, 2005a) to complete the algebraic analysis of families of frames and conduct a comparative study of the notion of independence of frames, so central in the theory of evidence, in an algebraic setup. The work is articulated into two parts.
In the first half we prove that families of frames are endowed with three

different algebraic structures, namely those of: 1. Boolean algebra, 2. upper semi-modular lattice, and 3. lower semi-modular lattice.
In the second part we study relationships and differences between the different forms of independence that can be introduced on such structures, and understand whether $\mathcal{IF}$ can be reduced to one of them.
The contribution of this work is then twofold. On one side, we complete the rich algebraic description of families of compatible frames by relating them to semi-modular lattices and matroids, extending some recent preliminary results (Cuzzolin, 2005a). On the other, we pose the notion of independence of frames in a wider context by highlighting its relation with classical independence in modern algebra. The overall picture is intriguing. Even though $\mathcal{IF}$ turns out not to be a cryptomorphic form of matroidal independence, it possesses interesting relations with several extensions of matroidal independence to lattices, stressing the need for a more general, comprehensive definition of this widespread and important notion.

1.2. Paper outline

We start in Section 2 by introducing the definitions of compatible frames and independence of frames as Boolean sub-algebras. We there recall a recent result linking frame independence and combinability with respect to Dempster's sum of belief functions. Combinability can then be studied in an algebraic setup by analyzing the algebraic properties of independence of frames.
In Section 3 the notion of independence on matroids is recalled. Even though families of frames endowed with $\mathcal{IF}$ *do not* form a matroid, matroids are strictly related to other algebraic structures like semi-modular lattices which *do* describe collections of compatible frames.
In Section 4 we prove indeed that families of frames are both upper and lower semi-modular lattices, according to the order relation we pick. On such structures matroidal independence can be extended, yielding several different relations whose meaning we thoroughly discuss and whose links with $\mathcal{IF}$ we highlight in Section 5. We conclude (Section 6) by showing that the binary (2-element) frames of a family are independent as Boolean sub-algebras iff they are *not* independent as elements of the corresponding matroid.

2. Independence of frames and Dempster's combination

In the theory of evidence (Shafer, 1976; Dempster, 1967), the mathematical representation of subjective probability is not a standard probability measure, but a *belief function.* Belief functions are defined by

distributing non-zero masses to elements of the power set of the domain or *frame* (rather then elements of the domain itself), masses that need to normalize to 1. In this interpretation they can be considered as finite *random sets* (Nguyen, 1978). Obviously, they include probability measures as the special case in which masses are given only to singletons. They can also be interpreted as lower bounds to an entire convex set of probability measures (Cuzzolin, 2008b).
Different sources of evidence on the same problem can generated two or more distinct belief functions. These functions have then to be merged to take into account the full available evidence. Several different operators have been proposed. Historically the first such proposal is due to Dempster (Dempster, 1968) (Section 2.1).
Such belief functions might not be defined on the same domain, but on different domains which nevertheless all relate to the same decision or estimation problem. This idea is encoded by the notion of *family of compatible frames* (Section 2.2). It turns out that belief functions defined on different compatible frames are guaranteed to be combinable if and only if such frames are *independent* is a way derived from Boolean algebras (Section 2.3). Even though not equivalent to independence of sources in the original formulation of Dempster's combination (Section 2.4), independence of frames is then strictly intertwined with combinability.

2.1. Dempster's combination of belief functions

A *basic probability assignment* (b.p.a.) over a finite set or *frame* (Shafer, 1976) Θ is a function $m : 2^\Theta \rightarrow [0, 1]$ on its power set $2^\Theta = \{A \subseteq \Theta\}$ such that $m(\emptyset) = 0$, $\sum_{A\subseteq\Theta} m(A) = 1$, $m(A) \geq 0 \ \forall A \subseteq \Theta$. The *belief function* (b.f.) $b : 2^\Theta \rightarrow [0, 1]$ associated with a basic probability assignment m on Θ is defined as $b(A) = \sum_{B\subseteq A} m(B)$. In more formal terms, a belief function b is the sum function associated with a basic probability assignment m_b on the partially ordered set $(2^\Theta, \subseteq)$, and m_b is its Moebius inverse (Aigner, 1979).

Different belief functions representing different pieces of evidence on the same problem Θ can be combined through an operator called Dempster's orthogonal sum.

Definition 1. *The* orthogonal sum *or* Dempster's sum *of two belief functions* $b_1, b_2 : 2^\Theta \rightarrow [0, 1]$ *is a new belief function* $b_1 \oplus b_2 : 2^\Theta \rightarrow [0, 1]$ *with b.p.a.*

$$m_{b_1\oplus b_2}(A) = \frac{\sum_{B\cap C=A} m_{b_1}(B)m_{b_2}(C)}{\sum_{B\cap C\neq\emptyset} m_{b_1}(B)m_{b_2}(C)}, \tag{1}$$

where m_{b_i} *denotes the b.p.a. of* b_i.

When the denominator of Equation (1) is zero the two functions are said to be *non-combinable*, and their orthogonal sum simply does not exist.
What happens when the belief functions to combine are defined on different frames? If such frames are *compatible* (in a way that was given a precise mathematical characterization by Shafer (Shafer, 1976)) then such a combination is still possible.

2.2. Families of frames

Given two frames (finite sets) Θ and Θ', a map $\rho : 2^{\Theta} \to 2^{\Theta'}$ is a *refining* if it maps the elements of Θ to a disjoint partition of Θ':

$$\rho(\{\theta\}) \cap \rho(\{\theta'\}) = \emptyset \ \ \forall \theta, \theta' \in \Theta; \qquad \bigcup_{\theta \in \Theta} \rho(\{\theta\}) = \Theta',$$

with $\rho(A) \doteq \cup_{\theta \in A} \rho(\{\theta\}) \ \forall A \subseteq \Theta$. The frame Θ' is called a *refinement* of Θ, Θ a *coarsening* of Θ'.
Shafer calls a structured collection $\mathcal{F}$ of frames a *family of compatible frames of discernment* ((Shafer, 1976), pages 121-125: see Appendix for the formal definition). In such a family, in particular, every pair of frames has a common refinement, i.e., a frame which is a refinement of both. Each finite collection of compatible frames has many such common refinements. One of these is particularly simple (Shafer, 1976).

Proposition 1. *If* $\{\Theta_1, ..., \Theta_n\}$ *are elements of a family of compatible frames* $\mathcal{F}$ *then there exists a* unique *common refinement* $\Theta \in \mathcal{F}$ *of them such that* $\forall \theta \in \Theta \ \exists \theta_i \in \Theta_i$ *for* $i = 1, ..., n$ *such that*

$$\{\theta\} = \rho_1(\{\theta_1\}) \cap \cdots \cap \rho_n(\{\theta_n\}), \tag{2}$$

where ρ_i *denotes the refining between* Θ_i *and* Θ.

This unique frame is called the *minimal refinement* $\Theta_1 \otimes \cdots \otimes \Theta_n$ of $\{\Theta_1, ..., \Theta_n\}$.

2.2.1. *Example of family of frames*

In the example of Figure 2.2.1 we want to find out the position of a target point in an image. We can pose the problem on a frame $\Theta_1 = \{c_1, ..., c_5\}$ obtained by partitioning the column range of the image into 5 intervals, or partition it into 10 intervals, yielding $\Theta_2 = \{c_{11}, c_{12}, ..., c_{51}, c_{52}\}$. The row range can also be divided in, say, 6 intervals $\Theta_3 = \{r_1, ..., r_6\}$.
All those frames belong to a family of compatible frames, with the collection of cells $\Theta = \{e_1, ..., e_{60}\}$ depicted in Figure 2.2.1-left as common refinement and refinings shown in Figure 2.2.1-right. It is easy to

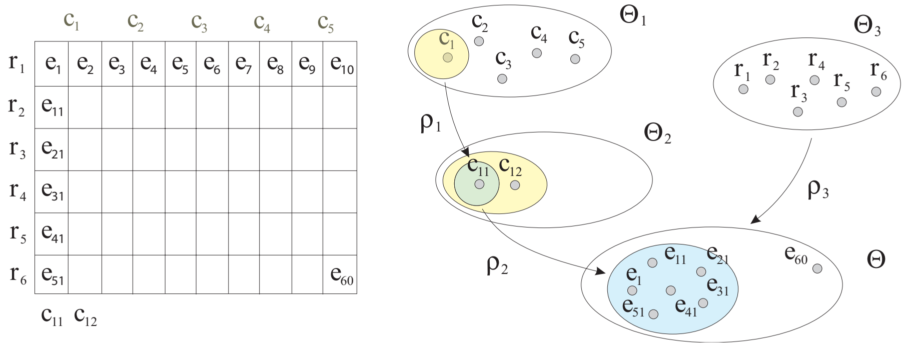

Figure 1. An example of family of compatible frames. Different discrete quantizations of row and column ranges of an image have as common refinement the set of cells on the left. The refinings ρ_1, ρ_2, ρ_3 between all those frames are shown on the right.

verify that Θ meets condition (2) for the frames Θ_2, Θ_3 as, for example, $\{e_{41}\} = \rho_2(c_{11}) \cap \rho_3(r_4)$. Hence Θ is the minimal refinement of Θ_2, Θ_3.

2.3. Independence of frames and Dempster's rule

The notion of compatible frames is crucial when facing real-world problems, as the available pieces of evidence are typically defined on different (even though related) domains. For instance, in computer vision inference on the objects present in a scene is typically made based on a number of salient measurements or *features* extracted from the image. These features can be of various nature and belong to different spaces. Nevertheless, they all say something on, for example, the class of the object we are observing.
When representing evidence as belief functions is then crucial to be able to combine them even when they belong to different frames of a family. It has been proven that this is guaranteed to be possible if and only if such frames are independent in the following way.

Definition 2. *Let $\Theta_1, ..., \Theta_n$ be elements of a family of compatible frames, and $\rho_i : \Theta_i \to 2^{\Theta_1 \otimes \cdots \otimes \Theta_n}$ the corresponding refinings to their minimal refinement.*
We say that $\{\Theta_1, ..., \Theta_n\}$ are independent *(*Shafer, 1976*) if*

$$\rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset \tag{3}$$

whenever $\emptyset \neq A_i \subseteq \Theta_i$ for $i = 1, ..., n$ (see Figure 2.3).

In particular, it is easy to see that if $\exists j \in [1, .., n]$ s.t. Θ_j is a coarsening of some other frame Θ_i, $|\Theta_j| > 1$, then $\{\Theta_1, ..., \Theta_n\}$ are

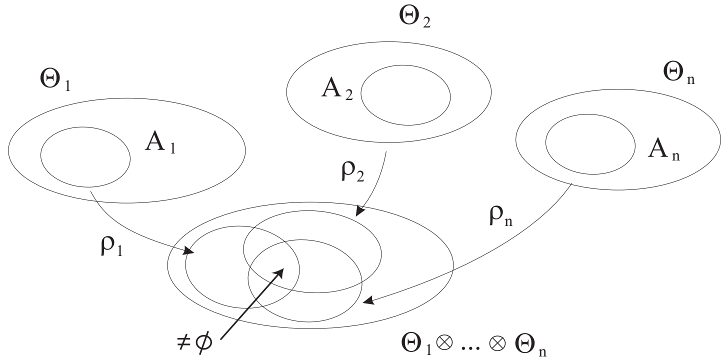

Figure 2. Independence of frames.

not independent.
Mathematically, families of compatible frames are collections of Boolean subalgebras of their common refinement ((Sikorski, 1964), see Appendix). Equation (3) is nothing but the independence condition for the associated sub-algebras.
We denote in the following by $\mathcal{IF}$ the independence relation introduced in Definition 2. It can be proven that an equivalent condition is (Cuzzolin, 2005a)

$$\Theta_1 \otimes \cdots \otimes \Theta_n = \Theta_1 \times \cdots \times \Theta_n, \tag{4}$$

i.e., their minimal refinement is their Cartesian product.

More importantly, belief functions defined on a number of frames $\{\Theta_1, ..., \Theta_n\}$ are surely combinable through Dempster's rule iff such frames are $\mathcal{IF}$ (Cuzzolin, 2005a).

Proposition 2. *Let $\{\Theta_1, ..., \Theta_n\}$ be a set of compatible frames. Then they are independent iff all the possible collections of b.f.s $b_1, ..., b_n$ defined on $\Theta_1, ..., \Theta_n$, respectively, are combinable through Dempster's sum (1) over their minimal refinement $\Theta_1 \otimes \cdots \otimes \Theta_n$.*

The notion of independence of frames is then intertwined with that of combinability in Dempster-Shafer theory.

2.4. Difference between independence of frames and independence of sources

Independence of frames $\mathcal{IF}$ as defined above is however distinct from the notion of *independence of sources* at the foundation of Dempster's combination.
The concept of belief function derives originally from a series of Dempster's works on the upper and lower probabilities induced by multivalued mappings (Dempster, 1967; Dempster, 1968; Dempster, 1969).

Each belief function on a certain frame Θ can be thought of as being induced by a probability distribution on another domain Ω, linked to the original frame by a map one-to-many (Shafer, 1990; Smets, 1987).

2.4.1. *Independence of sources in Dempster's sum*

Let us consider a problem in which we have probabilities for a question Q_1 and we want to derive a degree of belief for a related question Q_2, Ω and Θ the sets of possible answers to Q_1 and Q_2 respectively. So, given a probability measure P on Ω we seek for the belief $b(A)$ that $A \subseteq \Theta$ contains the correct response to Q_2 (see Figure 2.4.1). If we call $\Gamma(\omega)$

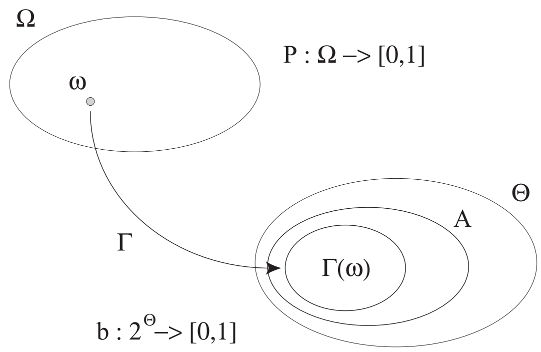

Figure 3. A probability measure P on Ω induces a belief function b on Θ through a multi-valued mapping Γ.

the subset of answers to Q_2 compatible with $\omega \in \Omega$, each element ω tells us that the answer to Q_2 is somewhere in A whenever $\Gamma(\omega) \subseteq A$. The map $\Gamma : \Omega \rightarrow 2^\Theta$ is called a *multi-valued mapping* from Ω to Θ.
The degree of belief $b(A)$ of an event $A \subseteq \Theta$ is then the total probability of all answers ω that satisfy the above condition, namely $b(A) = P(\{\omega | \Gamma(\omega) \subseteq A\})$.
Now let us consider two multi-valued mappings Γ_1, Γ_2 inducing two belief functions over a same frame Θ, Ω_1 and Ω_2 their domains and P_1, P_2 the probability measures over Ω_1 and Ω_2 respectively. If we suppose that the items of evidence generating P_1 and P_2 *independent*, we are allowed to build the product space $(\Omega_1 \times \Omega_2, P_1 \times P_2)$: detecting two outcomes $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ will then tell us that the answer to Q_2 is somewhere in $\Gamma_1(\omega_1) \cap \Gamma_2(\omega_2)$.
However, if this intersection is empty the two pieces of evidence are in contradiction. We then need to condition the product measure $P_1 \times P_2$ over the set of pairs (ω_1, ω_2) whose images have non-empty intersection,

namely

$$\begin{array}{l} \Omega = \{(\omega_1,\omega_2) \in \Omega_1 \times \Omega_2 | \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \neq \emptyset\}, \\ P = P_1 \times P_2|_\Omega, \qquad \Gamma(\omega_1,\omega_2) = \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2). \end{array} \quad (5)$$

The relation between the new belief function b induced by (5) and the pair of b.f.s being combined is exactly given by Dempster's rule (1).

Dempster's mechanism for evidence combination is then intimately connected to the assumption that the domains Ω_1, Ω_2 on which the evidence is present (in the form of a probability measure) are independent.

However, it is important to notice that this is not the same as independence of frames (Definition 2). While independence of sources is about the combination of belief functions on the *same* frame, $\mathcal{IF}$ is about the combination of belief functions living on *different* compatible frames. The notion of independence of frame is oblivious of the relationship between the original probability spaces that induced the belief functions to combine.

Let us see this in a simple counterexample.

2.4.2. *A counterexample*

Let $\Theta_1 = \{a, b\}$, $\Theta_2 = \{c, d\}$ be two compatible frames, and let ρ_i : $\Theta_i \rightarrow 2^{\Theta_1 \times \Theta_2}$ denote the corresponding refinings to their minimal refinement $\Theta = \Theta_1 \times \Theta_2$, i.e.,

$$\begin{array}{l} \rho_1(a) = \{(a,c),(a,d)\}, \ \rho_1(b) = \{(b,c),(b,d)\}, \\ \rho_2(c) = \{(a,c),(b,c)\}, \ \ \rho_2(d) = \{(a,d),(b,d)\}. \end{array}$$

The two frames Θ_1, Θ_2 are independent in the sense of Definition 2 ($\mathcal{IF}$).

Now, consider a belief function b_1 on Θ_1 induced by the multi-valued mapping $\Gamma_1 : \Omega_1 \rightarrow 2^\Theta_1$, and another belief function b_2 on Θ_2 induced by a second multi-valued mapping $\Gamma_2 : \Omega_2 \rightarrow 2^\Theta_2$. Let us denote by $\Omega_1 = \{\omega_1^1, \omega_1^2\}$, $\Omega_2 = \{\omega_2^1, \omega_2^2\}$ the two domains on which the original probability distributions inducing b_1 and b_2 are defined. Let us assume that such probability distributions P_1, P_2 are obtained by marginalizing on Ω_1 and Ω_2 the joint probability distribution P on $\Omega_1 \times \Omega_2$ given by

$$P(\omega_1^1,\omega_2^1) = \tfrac{1}{3}, \ P(\omega_1^1,\omega_2^2) = \tfrac{1}{3}, \ P(\omega_1^2,\omega_2^1) = \tfrac{1}{3}, \ P(\omega_1^2,\omega_2^2) = 0.$$

Therefore, P_1 and P_2 are given by

$$\begin{array}{l} P_1(\omega_1^1) = \tfrac{2}{3}, \ P_1(\omega_1^2) = \tfrac{1}{3}, \\ P_2(\omega_2^1) = \tfrac{2}{3}, \ P_2(\omega_2^2) = \tfrac{1}{3}. \end{array}$$

But it is easy to check that P is *not* the product measure of P_1 and P_2. Indeed

$$P(\omega_1^1, \omega_2^1) = \frac{1}{3} \neq P_1(\omega_1^1) \cdot P_2(\omega_2^1) = \frac{4}{9}.$$

Since $P \neq P_1 \times P_2$, the sources of information generating the belief functions b_1 and b_2 are *not* independent (in Dempster's sense), even though the related frames *are* independent (as Boolean sub-algebras).

3. An algebraic study of independence of frames

As it is strictly related to combinability in the original Dempster-Shafer theory of evidence, independence of frames plays an important role in the process of evidence aggregation in this framework for subjective probability.

An example is provided by the pose estimation problem in computer vision, which involves reconstructing the configuration of a moving body from an image sequence. In (Cuzzolin, 2005b) we presented a general framework for pose estimation of unknown objects based on Shafer's evidential reasoning, based on an *evidential model* of the object built in a training stage. Different image features were integrated to improve both estimation robustness and precision. All the measurements coming from one or more views were expressed as belief functions, and combined through Dempster's rule. If the frames representing distinct features are not independent, conflict may emerge and needs to be solved in order to produce a sensible estimate of the pose.

In (Cuzzolin, 2005a), starting from an analogy between independence of frames and linear independence, we conjectured a possible algebraic solution to the conflict problem, based on a mechanism similar to the classical Gram-Schmidt algorithm for the orthogonalization of vectors. Indeed, one can observe that the independence condition (3) resembles the condition under which a collection of vector spaces has maximal span (see also Equation (4)):

$$\begin{array}{ccc} v_1 + \cdots + v_n \neq 0, \ \forall v_i \in V_i & \equiv & span\{V_1, ..., V_n\} = V_1 \times \cdots \times V_n \\ \rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset, \ \forall A_i \subseteq \Theta_i & \equiv & \Theta_1 \otimes \cdots \otimes \Theta_n = \Theta_1 \times \cdots \times \Theta_n. \end{array} \quad (6)$$

Let us say that a number of subspaces $\{V_1, ..., V_n\}$ of a vector space V are independent iff each collection of vectors $\{v_i \in V_i, i = 1, ..n\}$ is linearly independent. It follows that while a number of compatible frames $\{\Theta_1, ..., \Theta_n\}$ are $\mathcal{IF}$ iff each selection of their representatives $A_i \in 2^{\Theta_i}$ has non-empty intersection, a collection of vectors subspaces $\{V_1, ..., V_n\}$ is independent iff for each choice of vectors $v_i \in V_i$ their sum is non-zero.

The collection of all the subspaces of a vector space or *projective geometry* forms a *modular lattice*. As we prove here, families of frames can be given the algebraic structure of *semi-modular lattice*, explaining in part this analogy. The goal of this paper is to go further and analyze the notion of independence of frames from an algebraic point of view, in order to understand whether it possesses any meaningful relations with other forms of independence.
The purpose is twofold: on one side, as independence of frames is formally independence of Boolean sub-algebras, this can be seen as a contribution to a better understanding of the notion in different fields of modern algebra. On the other hand, we provide the basis for an eventual algebraic proposal to the solution of the conflict problem in subjective probability.

The paradigm of abstract independence in modern algebra is represented by the notion of *matroid* (Section 3.1). Matroidal independence, though, extends to similar relations in other algebraic structures: in particular those of *semi-modular* and *geometric lattice* (Stern, 1999). Even though families of frames are *not* matroids (3.2), they form semi-modular lattices (Section 4) so that $\mathcal{IF}$ inherits interesting relations with some extensions of matroidal independence to semi-modular lattices (Section 5). Eventually, we will show how $\mathcal{IF}$ is in fact opposed to matroidal independence (Section 6).

3.1. Matroids

Matroids were introduced by Whitney in the Thirties (Whitney, 1935). He and other authors, among which van der Waerden (van der Waerden, 1937), Mac Lane (Lane, 1938), and Teichmuller (Teichmuller, 1936) recognized that several apparently different notions of dependence (Harary and Tutte, 1969; Beutelspacher and Rosenbaum, 1998) in algebra (circuits in graphs, flats in affine geometries) have many properties in common with linear dependence of vectors. In particular, matroids have been an important source for semi-modular lattices (Birkhoff, 1935). Let us briefly introduce the basic notions of matroid theory (Oxley, 1992).

Definition 3. *A* matroid $M = (E, \mathcal{I})$ *is a pair formed by a* ground set E*, and a collection of* independent sets $\mathcal{I} \subseteq 2^E$*, such that:*

1. $\emptyset \in \mathcal{I}$;
2. *if* $I \in \mathcal{I}$ *and* $I' \subseteq I$ *then* $I' \in \mathcal{I}$;
3. *if* I_1 *and* I_2 *are in* $\mathcal{I}$*, and* $|I_1| < |I_2|$*, then there is an element* e *of* $I_2 - I_1$ *such that* $I_1 \cup e \in \mathcal{I}$.

Condition (3) is called *augmentation* axiom, and is the foundation of the notion of matroidal independence.
The name "matroid" was coined by Whitney (Whitney, 1935) because of a fundamental class of matroids which arise form matrices. The collection of columns of a matrix together with the collection of linearly independent (in the ordinary sense) sets of columns form indeed a matroid, called *vector matroid*.
Consider as an example the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

with column labels $E = \{1, 2, 3, 4, 5\}$. Obviously the collection of independent sets in E is $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$.
It is interesting to see that linearly independent vectors in a vector space actually satisfy the augmentation axiom (3) of Definition 3. Let I_1 and I_2 be linearly independent subsets such that $|I_1| < |I_2|$. Let W be the subspace spanned by $I_1 \cup I_2$. Then $\dim W$ is at least $|I_2|$ (as I_2, the largest of the two collections, is linearly independent). Now suppose that $I_1 \cup \{e\}$ is linearly dependent for all $e \in I_2 \setminus I_1$. Then W is contained in the span of I_1, thus $|I_2| \leq \dim W \leq |I_1| < |I_2|$ which is a contradiction.

3.2. Families of frames are not matroids

It is natural to wonder whether $\mathcal{IF}$ could in fact be some form of matroidal independence, i.e. whether for each family $\mathcal{F}$ of compatible frames, $(\mathcal{F}, \mathcal{IF})$ is a matroid. Unfortunately this is not the case.

Theorem 1. *A family of compatible frames $\mathcal{F}$ endowed with Shafer's independence $\mathcal{IF}$ is* not *a matroid.*

Proof. In fact, $\mathcal{IF}$ does not meet the augmentation axiom (3) of Definition 3. Consider two independent compatible frames $I = \{\Theta_1, \Theta_2\}$. If we pick another arbitrary frame Θ_3 of the family, the collection $I' = \{\Theta_3\}$ is trivially $\mathcal{IF}$. Suppose $\Theta_3 \neq \Theta_1, \Theta_2$. Then, since $|I| > |I'|$, by augmentation we can form a new pair of independent frames by adding any of Θ_1, Θ_2 to Θ_3. But it is easy to find a counterexample, for instance by picking as Θ_3 the common coarsening of Θ_1 and Θ_2 (remember the remark after Definition 2). □

Independence of Boolean sub-algebras is then not independence in matroidal sense. Nevertheless independence exists in other forms defined on other algebraic structures. Two of them, in particular, inherit their own particular notion of independence from that of matroids. Both those structures are particular classes of *lattice*.

4. The lattice of frames

Collections of compatible frames (see Appendix) are collections of Boolean sub-algebras of (the power set of) their minimal refinement. In addition, as it has been proven in (Cuzzolin, 2005a), they possess the structure of *lattice*. Two different order relations between frames can be defined. According to the chosen ordering, the resulting lattice will be either *upper* or *lower* semi-modular.
This allows us to introduce a number of different extensions of matroidal independence to compatible frames, as we will see in Section 5.

4.1. Lattices

A *partially ordered set* or *poset* is a set P endowed with a binary relation $\leq$ such that, for all x, y, z in P the following conditions hold:

1. $x \leq x$;
2. if $x \leq y$ and $y \leq x$ then $x = y$;
3. if $x \leq y$ and $y \leq z$ then $x \leq z$.

In a poset we say that x *covers* y ($x \succ y$) if $x \geq y$ and there is no intermediate element in the chain (collection of consecutive elements) linking them.
A classical example is the power set 2^Θ of a set Θ together with the set-theoretic inclusion relation $\subset$. A poset has *finite length* if the length of all its chains is bounded.
Given two elements $x, y \in P$ of a poset P their *least upper bound* $\sup_P(x, y) = x \vee y$ is the smallest element of P that is bigger than both x and y. Their *greatest lower bound* $\inf_P(x, y) = x \wedge y$ is the biggest element of P that is smaller than both x and y. In the case of $L = (2^\Theta, \subset)$ "sup" is the usual set-theoretic union, $A \vee B = A \cup B$, while "inf" is the usual intersection $A \wedge B = A \cap B$.
By induction, sup and inf can be defined for arbitrary finite collections too. However, not any pair of elements of a poset, in general, is guaranteed to admit inf and/or sup.

Definition 4. *A* lattice *L is a poset in which* each pair *of elements admits both* inf *and* sup.

When each *arbitrary* (even not finite) collection of elements of L admits both inf and sup, L is said *complete*. In this case there exist $\mathbf{0} \equiv \wedge L$, $\mathbf{1} \equiv \vee L$ called respectively *initial* and *final* element of L. 2^Θ is complete, with $\mathbf{0} = \emptyset$ and $\mathbf{1} = \{\Theta\}$. The *height* $h(x)$ of an element x in L is the length of the maximal chain from $\mathbf{0}$ to x. For the power set 2^Θ, the height of a subset $A \in 2^\Theta$ is simply its cardinality $|A|$.

4.2. Families of frames as lattices

In a family of compatible frames one can define two distinct order relations on pairs of frames, both associated with the notion of refining (Section 2.2):

$$\Theta_1 \leq^* \Theta_2 \Leftrightarrow \exists \rho : 2^{\Theta_1} \to 2^{\Theta_2} \ refining \tag{7}$$

(Θ_1 is a coarsening of Θ_2), or

$$\Theta_1 \leq \Theta_2 \Leftrightarrow \exists \rho : 2^{\Theta_2} \to 2^{\Theta_1} \ refining \tag{8}$$

i.e., Θ_1 is a refinement of Θ_2. Relation (8) is clearly the inverse of (7). It makes sense to distinguish them explicitly as they generate two distinct algebraic structures, in turn associated with different extensions of the notion of matroidal independence, as we will see in Section 5.
As it has been proven in (Cuzzolin, 2005a), a family of frames $\mathcal{F}$ is a poset with respect to both (7) and (8). More precisely, after introducing the notion of *maximal coarsening* as the largest cardinality common coarsening $\Theta_1 \oplus \cdots \oplus \Theta_n$ of a given collection of frames $\Theta_1, \cdots, \Theta_n$, we can prove that (Cuzzolin, 2005a)

Proposition 3. *Both* $(\mathcal{F}, \leq)$ *and* $(\mathcal{F}, \leq^*)$ *where* $\mathcal{F}$ *is a family of compatible frames of discernment are lattices, where*

$$\bigwedge_i \Theta_i = \bigotimes_i \Theta_i, \ \ \bigvee_i \Theta_i = \bigoplus_i \Theta_i,$$
$$\bigwedge_i^* \Theta_i = \bigoplus_i \Theta_i, \ \ \bigvee_i^* \Theta_i = \bigotimes_i \Theta_i.$$

4.3. Upper and lower semi-modularity

A special class of lattices (*modular* lattices[3]) arises from *projective geometries*, i.e. collections $L(V)$ of all subspaces of a vector space V. Modular lattices, as many authors have shown, are also related to abstract independence. This quality is retained by a wider class of lattices called *semi-modular* lattices.

[3] A lattice L is modular iff if $y \leq x$, $x \wedge z = y \wedge z$, $x \vee z = y \vee z$ then $x = y$.

Definition 5. *A lattice L is* upper semi-modular *if for each pair x, y of elements of L, $x \succ x \wedge y$ implies $x \vee y \succ y$. A lattice L is* lower semi-modular *if for each pair x, y of elements of L, $x \vee y \succ y$ implies $x \succ x \wedge y$.*

Clearly if L is upper semi-modular with respect to an order relation $\leq$, than the corresponding dual lattice with order relation $\leq^*$ is lower semi-modular, as

$$x \succ x \wedge y \vdash x \vee y \succ y \quad \Rightarrow \quad x \vee^* y \succ^* x \vdash y \succ^* x \wedge^* y. \qquad (9)$$

For lattices of finite length, upper and lower semi-modularity together imply modularity. In this sense semi-modularity is indeed "one half" of modularity.

4.4. Upper and lower semi-modular lattices of frames

Families of frames possess indeed the structure of semi-modular lattice.

Theorem 2. $(\mathcal{F}, \leq)$ *is an upper semi-modular lattice;* $(\mathcal{F}, \leq^*)$ *is a lower semi-modular lattice.*

Proof. We just need to prove the upper semi-modularity with respect to $\leq$.
Consider two compatible frames Θ, Θ', and suppose that Θ covers their minimal refinement $\Theta \otimes \Theta'$ (their inf with respect to $\leq$). The proof articulates into the following steps (see Figure 4.4):

– as Θ covers $\Theta \otimes \Theta'$ we have that $|\Theta| = |\Theta \otimes \Theta'| + 1$;

– this means that there exists a single element $p \in \Theta$ which is refined into a pair of elements $\{p_1, p_2\}$ of $\Theta \otimes \Theta'$, while all other elements of Θ are left unchanged: $\{p_1, p_2\} = \rho(p)$;

– this in turn implies that p_1, p_2 each belong to the image of a different element of Θ' (otherwise Θ would itself be a refinement of Θ', and we would have $\Theta \otimes \Theta' = \Theta$):

$$p_1 \in \rho'(p'_1), \qquad p_2 \in \rho'(p'_2);$$

– now, if we merge p'_1, p'_2 we obviously have a coarsening Θ'' of Θ':

$$\{p'_1, p'_2\} = \rho''(p'');$$

– but Θ'' is a coarsening of Θ, too, as we can build the refining $\sigma : \Theta'' \to 2^{\Theta}$:

$$\sigma(q) = \rho'(\rho''(q))$$

where $\rho'(\rho''(q))$ is a subset of Θ $\forall q \in \Theta''$:

- if $q = p''$, $\sigma(q)$ is $\{p\} \cup (\rho'(p'_1) \setminus \{p_1\}) \cup (\rho'(p'_2) \setminus \{p_2\})$;
- if $q \neq p''$, $\rho'(\rho''(q))$ is also a set of elements of Θ, as all elements of Θ but p are left unchanged by ρ.

– as $|\Theta''| = |\Theta'| - 1$ we have that Θ'' is the maximal coarsening of Θ, Θ': $\Theta'' = \Theta \oplus \Theta'$;

– hence Θ' covers $\Theta \oplus \Theta'$, which is the sup of Θ, Θ' in $(\mathcal{F}, \leq)$.

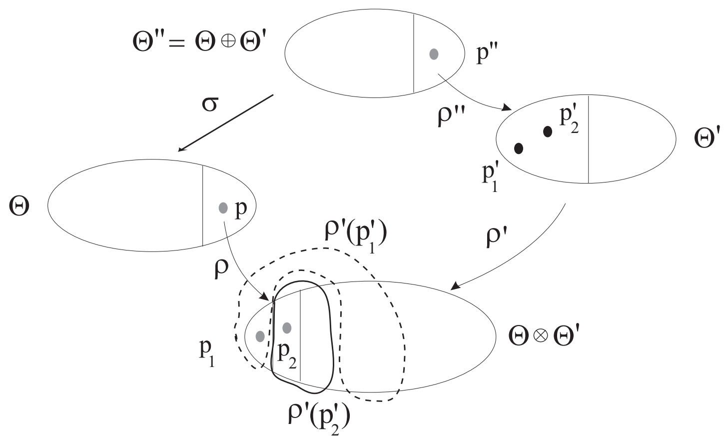

Figure 4. Proof of the upper semi-modularity of $(\mathcal{F}, \leq)$.

The lower semi-modularity with respect to $\leq^*$ comes immediately from (9). □

Theorem 2 strengthens the main result of (Cuzzolin, 2005a), where we proved that finite families of frames are Birkhoff. A lattice is *Birkhoff* if $x \wedge y \prec x, y$ implies $x, y \prec x \vee y$. (Upper) semi-modularity implies the Birkhoff property, but not vice-versa.

We will here focus on *finite* families of frames. Given a set of compatible frames $\{\Theta_1, ..., \Theta_n\}$ we can consider the set $P(\Theta)$ of all partitions of their minimal refinement $\Theta = \Theta_1 \otimes \cdots \otimes \Theta_n$. As the independence condition (Definition 2) involves only partitions of $\Theta_1 \otimes \cdots \otimes \Theta_n$, we can conduct our analysis there. We denote by $L(\Theta) \doteq (P(\Theta), \leq)$, $L^*(\Theta) \doteq (P(\Theta), \leq^*)$ the two lattices associated with the set $P(\Theta)$ of partitions of Θ, with order relations (7), (8) respectively.

4.4.1. *Example: the partition lattice P_4*

Consider for example the partition lattice associated with a frame of size 4: $\Theta = \{1, 2, 3, 4\}$, depicted in Figure 4.4.1, with order relation $\leq^*$. Each edge indicates here that the bottom partition covers $\succ$ the top

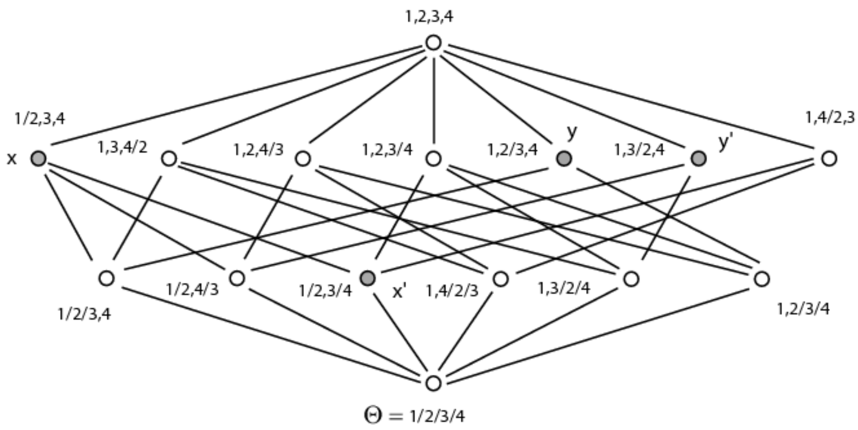

Figure 5. The partition (lower) semi-modular lattice $L^*(\Theta)$ for a frame Θ of size 4. Partitions $A_1, ..., A_k$ of Θ are denoted by $A_1/.../A_k$. Partitions with the same number of elements are arranged on the same level. An edge between two nodes indicates that the bottom partition covers the top one.

one.
To understand how inf and sup work in the frame lattice, pick the partitions

$$x = \{1/2, 3, 4\}, \quad x' = \{1/2, 3/4\}.$$

According to the diagram the partition $x \vee^* x'$ which refines both and has smallest size is $\Theta = \{1/2/3/4\}$ itself. Their inf $x \wedge^* x'$ is x, as x' is a refinement of x.
If we pick instead the pair of partitions $y = \{1, 2/3/4\}$ and $y' = \{1, 3/2/4\}$, we can notice that both y, y' cover their inf $y \wedge^* y' = \{1, 2, 3, 4\}$ but in turn their sup $y \vee^* y' = \Theta = \{1/2/3/4\}$ does not cover them. Therefore, $(P(\Theta), \leq^*)$ is not upper semi-modular but *lower* semi-modular.

4.4.2. *Independence on semi-modular and geometric lattices*

On the *atoms* of a lattice, i.e. the elements of the lattice covering **0** (think of one-dimensional subspaces of a vector space V, Figure 4.4.2) it is possible to define a matroidal independence relation. In particular, for each upper semi-modular lattice L there exists a collection $\mathcal{I}$ of sets of atoms such that $(A, \mathcal{I})$ is a matroid. As families of frames form semi-modular lattices, we can explore the relations between $\mathcal{IF}$ and several extensions of matroidal independence to both $L(\Theta)$ and $L^*(\Theta)$.

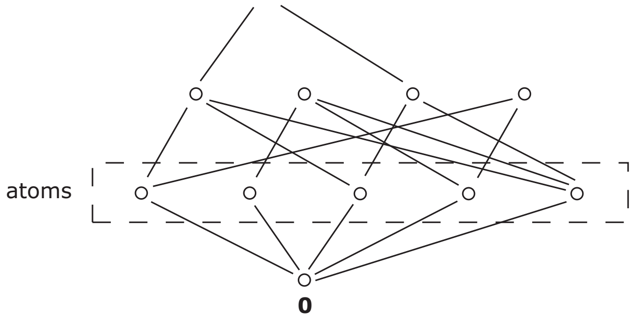

Figure 6. A lattice can be represented as a (*Hasse*) diagram in which covering relations are drawn as undirected edges. The atoms of a lattice which initial element **0** (bounded below) are the elements covering **0**.

5. Independence on lattices and independence of frames

5.1. Atom matroid of a semi-modular lattice

Consider again the classical example of linear independence of vectors. By definition $\{v_1, ..., v_n\}$ are *linearly independent* iff

$$\sum_i \alpha_i v_i = 0 \quad \vdash \quad \alpha_i = 0 \; \forall i.$$

This classical definition can though be given several equivalent formulations:

$$\begin{array}{lll} \mathcal{I}_1: & v_j \not\subset span(v_i, i \neq j) & \forall j = 1, ..., n; \\ \mathcal{I}_2: & v_j \cap span(v_1, ..., v_{j-1}) = 0 & \forall j = 2, ..., n; \\ \mathcal{I}_3: & \dim(span(v_1, ..., v_n)) = n. & \end{array} \tag{10}$$

Remember that the one-dimensional subspaces of a vector space V are the atoms of the lattice $L(V)$ all all the linear subspaces of V, for which $span = \vee$, $\cap = \wedge$, $dim = h$ and $\mathbf{0} = 0$. Following this intuition we can extend the relations (10) to collections of arbitrary (non necessarily atomic) non-zero elements of an arbitrary semi-modular lattice with initial element.

Definition 6. *The following relations on the elements of a semi-modular lattice with initial element* **0** *can be defined:*

1. $\{l_1, ..., l_n\}$ *are* $\mathcal{I}_1$ *if*

$$l_j \not\leq \bigvee_{i \neq j} l_i \;\; \equiv \;\; l_j \wedge \bigvee_{i \neq j} l_i \neq l_j \quad \forall j = 1, ..., n;$$

2. $\{l_1, ..., l_n\}$ *are* $\mathcal{I}_2$ *if*

$$l_j \wedge \bigvee_{i<j} l_i = \mathbf{0} \quad \forall j = 2, ..., n;$$

3. $\{l_1, ..., l_n\}$ *are* $\mathcal{I}_3$ *if*

$$h(\bigvee_i l_i) = \sum_i h(l_i).$$

Graphical interpretations of those relations in terms of Hasse diagrams are given in Figure 5.1. The have been studied by several authors in the past: our purpose here is to understand their relation with independence of frames in the semi-modular lattice of frames.
In the general case $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ are distinct, and none of them originates

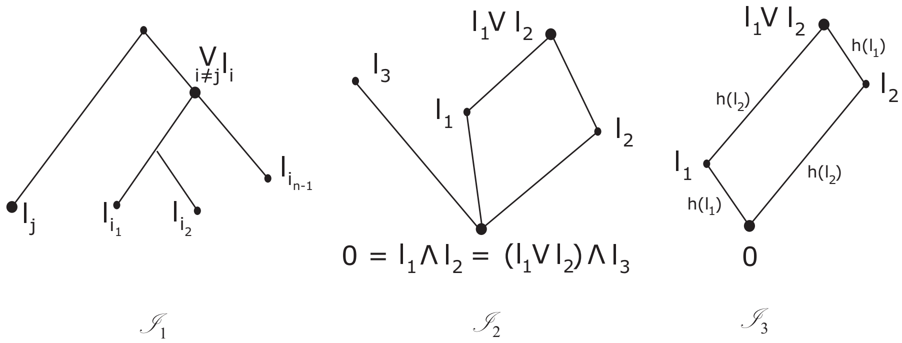

Figure 7. Graphical interpretation of the relations introduced in Definition 6.

a matroid (see Appendix). However, when defined on the *atoms* of an upper semi-modular lattice with initial element they do coincide, and form a matroid (Szasz, 1963).

Proposition 4. *The restrictions of the above relations to the set of the atoms* A *of an upper semi-modular lattice* L *with initial element are equivalent, namely* $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3 = \mathcal{I}$ *on* A*, and* $(A, \mathcal{I})$ *is a matroid.*

As the partition lattice has both an upper $L(\Theta)$ and lower $L^*(\Theta)$ semi-modular form, we can introduce two forms $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_1^*$, $\mathcal{I}_2^*$, $\mathcal{I}_3^*$ of the above relations associated with $L(\Theta)$ and $L^*(\Theta)$, respectively. These constitute different valid extensions of matroidal independence to elements of semi-modular lattices. As families of compatible frames form semi-modular lattices in their own right, it is natural to wonder what is their relation, if any, with Shafer's independence of frames $\mathcal{IF}$.

5.2. Boolean and lattice independence in the upper semi-modular lattice $L(\Theta)$

5.2.1. *Form of the relations*

In $L(\Theta)$ the relations introduced in Definition 6 assume the forms:

$$\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_1 \quad \Leftrightarrow \quad \Theta_j \otimes \bigoplus_{i \neq j} \Theta_i \neq \Theta_j \quad \forall\, j = 1, ..., n, \tag{11}$$

$$\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_2 \quad \Leftrightarrow \quad \Theta_j \otimes \bigoplus_{i < j} \Theta_i = \Theta \quad \forall\, j = 2, ..., n, \tag{12}$$

$$\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_3 \quad \Leftrightarrow \quad |\Theta| - \Big|\bigoplus_{i=1}^n \Theta_i\Big| = \sum_{i=1}^n (|\Theta| - |\Theta_i|), \tag{13}$$

as in the lattice $L(\Theta)$ we have $\Theta_i \wedge \Theta_j = \Theta_i \otimes \Theta_j$, $\Theta_i \vee \Theta_j = \Theta_i \oplus \Theta_j$, $h(\Theta_i) = |\Theta| - |\Theta_i|$, and $\mathbf{0} = \Theta$.

They read as follows: $\{\Theta_1, ..., \Theta_n\}$ are $\mathcal{I}_1$ iff no frame Θ_j is a refinement of the maximal coarsening of all the others. They are $\mathcal{I}_2$ iff $\forall\, j = 2, ..., n$ Θ_j does not have a non-trivial common refinement with the maximal coarsening of all its predecessors.

The interpretation of $\mathcal{I}_3$ is perhaps more interesting. The latter is equivalent to say that the coarsening that generates $|\bigoplus_{i=1}^n \Theta_i|$ can be broken up into n steps of the same length of the coarsenings that generates each of the frames Θ_i starting from Θ: First Θ_1 is obtained from Θ by merging $|\Theta| - |\Theta_1|$ elements, then $|\Theta| - |\Theta_2|$ elements of this new frame are merged, and so on until we get $|\bigoplus_{i=1}^n \Theta_i|$. We will return on this when discussing the dual relation on the lower semi-modular lattice $L^*(\Theta)$.

To study the logical implications between these lattice-theoretic relations and independence of frames, and between themselves, we first need an interesting lemma.

Lemma 1. *$\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF}$, $n > 1 \vdash \bigoplus_{i=1}^n \Theta_i = \mathbf{0}_{\mathcal{F}}$.*

Proof. We prove Lemma 1 by induction. For $n = 2$, let us suppose that $\{\Theta_1, \Theta_2\}$ are $\mathcal{IF}$. Then $\rho_1(A_1) \cap \rho_2(A_2) \neq \emptyset \ \forall A_1 \subseteq \Theta_1, A_2 \subseteq \Theta_2$, $A_1, A_2 \neq \emptyset$ (ρ_i denotes as usual the refining from Θ_i to $\Theta_1 \otimes \Theta_2$). Suppose by absurd that their common coarsening contains more than a single element, $\Theta_1 \oplus \Theta_2 = \{a, b\}$. But then

$$\rho_1(\rho^1(a)) \cap \rho_2(\rho^2(b)) = \emptyset$$

(where ρ^i denotes the refining between $\Theta_1 \oplus \Theta_2$ and Θ_i), going against the hypothesis.

Induction step. Suppose that the thesis is true for $n-1$. We know that

$\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF}$ implies $\{\Theta_i, i \neq j\} \in \mathcal{IF}$. By inductive hypothesis, the latter implies:

$$\bigoplus_{i \neq j} \Theta_i = \mathbf{0}_{\mathcal{F}} \quad \forall j = 1, ..., n.$$

Of course then, since $\mathbf{0}_{\mathcal{F}}$ is a coarsening of $\Theta_j \ \forall \ j$,

$$\Theta_j \oplus \bigoplus_{i \neq j} \Theta_i = \Theta_j \oplus \mathbf{0}_{\mathcal{F}} = \mathbf{0}_{\mathcal{F}}.$$

□

Let us work our way up by considering situations of increasing generality.

5.2.2. *Atomic case*

The case of the atoms of $L(\Theta)$ (the frames of cardinality $|\Theta| - 1$) is in fact trivial. By Proposition 4, $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3 = \mathcal{I}$. On the other side,

Theorem 3. *Collections of atoms of $L(\Theta)$ are never $\mathcal{IF}$.*

Proof. It comes from the fact that their minimal refinement can only be Θ, whose cardinality is way smaller than $(n-1) \cdot ... \cdot (n-1)$ (equivalent condition for $\mathcal{IF}$, Equation (4)). □

5.2.3. *Pairs of frames*

Let us consider next the special case of collections of just two frames. For $n = 2$ the three relations $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ read respectively as

$$\Theta_1 \otimes \Theta_2 \neq \Theta_1, \Theta_2, \quad \Theta_1 \otimes \Theta_2 = \Theta, \quad |\Theta| + |\Theta_1 \oplus \Theta_2| = |\Theta_1| + |\Theta_2|. \tag{14}$$

It is interesting to remark that $\{\Theta_1, \Theta_2\} \in \mathcal{I}_1 \vdash \Theta_1, \Theta_2 \neq \Theta$. We can prove the following logical implications.

Theorem 4. *1. $\{\Theta_1, \Theta_2\} \in \mathcal{IF} \vdash \{\Theta_1, \Theta_2\} \in \mathcal{I}_1$ if $\Theta_1, \Theta_2 \neq \mathbf{0}_F$;*

2. $\{\Theta_1, \Theta_2\} \in \mathcal{I}_1 \not\vdash \{\Theta_1, \Theta_2\} \in \mathcal{IF}$;

3. $\{\Theta_1, \Theta_2\} \in \mathcal{I}_2 \vdash \{\Theta_1, \Theta_2\} \in \mathcal{I}_1$ iff $\Theta_1, \Theta_2 \neq \Theta$;

4. $\{\Theta_1, \Theta_2\} \in \mathcal{I}_3 \vdash \{\Theta_1, \Theta_2\} \in \mathcal{I}_1$ iff $\Theta_1, \Theta_2 \neq \Theta$;

5. $\{\Theta_1, \Theta_2\} \in \mathcal{IF} \wedge \mathcal{I}_3$ iff $\Theta_i = \mathbf{0}_F$ and $\Theta_j = \Theta$.

Proof. 1. If $\{\Theta_1, \Theta_2\}$ are $\mathcal{IF}$ then Θ_1 is not a refinement of Θ_2, and vice-versa, unless one of them is $\mathbf{0}_{\mathcal{F}}$. But then they are $\mathcal{I}_1$ ($\Theta_1 \otimes \Theta_2 \neq \Theta_1, \Theta_2$).

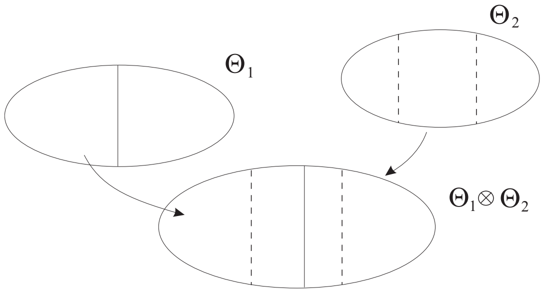

Figure 8. Counterexample for the conjecture $\mathcal{I}_1 \vdash \mathcal{IF}$ of Theorem 4.

2. We can give a counterexample as in Figure 2 in which $\{\Theta_1, \Theta_2\}$ are $\mathcal{I}_1$ (as none of them is refinement of the other one) but their minimal refinement $\Theta_1 \otimes \Theta_2$ has cardinality $4 \neq |\Theta_1| \cdot |\Theta_2| = 6$ (hence they are not $\mathcal{IF}$).

3. Obvious.

4. $\mathcal{I}_3 \vdash \mathcal{I}_1$ is equivalent to $\neg\mathcal{I}_1 \vdash \neg\mathcal{I}_3$. But $\{\Theta_1, \Theta_2\} \in \neg\mathcal{I}_1$ reads as $\Theta_1 \otimes \Theta_2 = \Theta_i$, which is in turn equivalent to $\Theta_1 \oplus \Theta_2 = \Theta_j$. I.E.,
$$\{\Theta_1, \Theta_2\} \in \mathcal{I}_3 \equiv |\Theta| + |\Theta_j| = |\Theta_i| + |\Theta_j| \equiv |\Theta| = |\Theta_i|.$$
But then $\{\Theta_1, \Theta_2\} \in \mathcal{I}_3 \vdash \{\Theta_1, \Theta_2\} \in \mathcal{I}_1$ iff $\Theta_1, \Theta_2 \neq \Theta$.

5. As $\{\Theta_1, \Theta_2\}$ are $\mathcal{IF}$, $|\Theta_1 \oplus \Theta_2| = 1$ so that $\{\Theta_1, \Theta_2\} \in \mathcal{I}_3 \equiv |\Theta| + 1 = |\Theta_1| + |\Theta_2|$. Now, by definition
$$|\Theta| \geq |\Theta_1 \otimes \Theta_2| = |\Theta_1||\Theta_2|$$
(the last passage holding as those frames are $\mathcal{IF}$). Therefore $\{\Theta_1, \Theta_2\} \in \mathcal{IF}$ and $\{\Theta_1, \Theta_2\} \in \mathcal{I}_3$ together imply
$$|\Theta_1| + |\Theta_2| \geq |\Theta_1||\Theta_2| + 1 \equiv |\Theta_1| - 1 \geq |\Theta_1||\Theta_2| - |\Theta_2| = |\Theta_2|(|\Theta_1| - 1) \equiv |\Theta_2| \leq 1,$$
which holds iff the equality holds, i.e., $\Theta_2 = \mathbf{0}_{\mathcal{F}}$. The latter implies $\Theta = \Theta_1 \otimes \Theta_2 = \Theta_1$.

□

In the singular case $\Theta_1 = \mathbf{0}_{\mathcal{F}}, \Theta_2 = \Theta$, by definition (14) the pair $\{\mathbf{0}_{\mathcal{F}}, \Theta\}$ is both $\mathcal{I}_2$ and $\mathcal{I}_3$, but not $\mathcal{I}_1$. Besides, two frames can be both $\mathcal{I}_2$ and $\mathcal{IF}$ without being singular in the above sense. The pair of frames $\{y, y'\}$ in Figure 4.4.1 provides such an example, as $y \otimes y' = \Theta$ ($\mathcal{I}_2$) and they are $\mathcal{IF}$.
As it well known that (Szasz, 1963) on an upper-semi-modular lattice (like $L(\Theta)$)

Proposition 5. $\mathcal{I}_3 \vdash \mathcal{I}_2$.

the overall picture formed by the different lattice-extended matroidal independence relations for *pairs* of frames is as in Figure 5.2.3. Independence

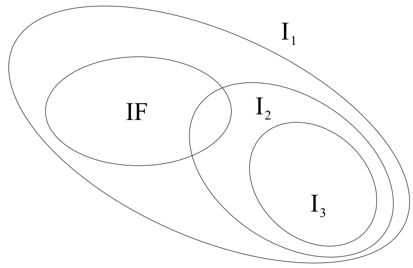

Figure 9. Relations between independence of frames $\mathcal{IF}$ and different extensions of matroidal independence to pairs of frames as elements of the upper semi-modular lattice $L(\Theta)$.

of frames and the most demanding form $\mathcal{I}_3$ of extended matroidal independence relation to frames as elements of an upper semi-modular lattice are mutual exclusive, and are both stronger than the weakest form $\mathcal{I}_1$. Some of those features are retained by the general case too.

5.2.4. *General case, $n > 2$*
The situation is somehow different in the general case of a collection of n frames. $\mathcal{IF}$ and $\mathcal{I}_1$, in particular, turn out to be incompatible.

Theorem 5. *If $\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF}$, $n > 2$ then $\{\Theta_1, ..., \Theta_n\} \in \neg\mathcal{I}_1$.*

Proof. If $\{\Theta_1, ..., \Theta_n\}$ are $\mathcal{IF}$ then any collection formed by some of those frames is $\mathcal{IF}$ (otherwise we could find empty intersections in $\Theta_1 \otimes \cdots \otimes \Theta_n$).
But then, by Lemma 1,

$$\bigoplus_{i \in L \subset \{1,...,n\}} \Theta_i = \mathbf{0}_{\mathcal{F}}$$

for all subsets L of $\{1, ..., n\}$ with at least 2 elements: $|L| > 1$.
But then, as $L = \{i \neq j, i \in \{1, ..., n\}\}$ has cardinality $n - 1 > 1$ (as $n > 2$) we have $\bigoplus_{i \neq j} \Theta_i = \mathbf{0}_{\mathcal{F}}$ for all $j \in \{1, ..., n\}$. Therefore

$$\Theta_j \otimes \bigoplus_{i \neq j} \Theta_i = \Theta_j \otimes \mathbf{0}_{\mathcal{F}} = \Theta_j \quad \forall j \in \{1, ..., n\}$$

and $\{\Theta_1, ..., \Theta_n\}$ are not $\mathcal{I}_1$. □

Theorem 6. *If $\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF}$, $n > 2$ then $\{\Theta_1, ..., \Theta_n\} \in \neg\mathcal{I}_2$.*

Proof. If $\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF}$ then $\{\Theta_1, ..., \Theta_{k-1}\} \in \mathcal{IF}$ for all $k = 3, ..., n$. But by Lemma 1 this implies $\bigoplus_{i<k} \Theta_i = \mathbf{0}_{\mathcal{F}}$, so that:

$$\Theta_k \otimes \bigoplus_{i<k} \Theta_i = \Theta_k \quad \forall k > 2.$$

Now, $\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF}$ with $n > 2$ implies $\Theta_k \neq \Theta$ $\forall k$. That holds because, as $n > 2$, there is at least one frame Θ_i in the collection $\Theta_1, ..., \Theta_n$ distinct from $\mathbf{0}_{\mathcal{F}}$, and clearly $\{\Theta_i, \Theta\}$ are not $\mathcal{IF}$ (as Θ_i is a non-trivial coarsening of Θ). Hence

$$\Theta_k \otimes \bigoplus_{i<k} \Theta_i \neq \Theta \quad \forall k > 2,$$

which is in fact a much stronger condition than $\neg\mathcal{I}_2$. □

A special case is that in which one of the frames is Θ itself. By Definitions (11) and (12) of $\mathcal{I}_1$ and $\mathcal{I}_2$, if $\exists j : \Theta_j = \Theta$ then $\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_2$ $\vdash \{\Theta_1, ..., \Theta_n\} \in \neg\mathcal{I}_1$. From Proposition 5 it follows that

Corollary 1. *If* $\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF}$, $n > 2$ *then* $\{\Theta_1, ..., \Theta_n\} \in \neg\mathcal{I}_3$.

Putting together the results of Theorems 3,4 and 6 and Corollary 1 we get that $\mathcal{IF}$ and $\mathcal{I}_3$ are incompatible in all significant cases.

Corollary 2. *If* $\{\Theta_1, ..., \Theta_n\}$ *are* $\mathcal{IF}$ *then they are not* $\mathcal{I}_3$, *unless* $n = 2$, $\Theta_1 = \mathbf{0}_{\mathcal{F}}$ *and* $\Theta_2 = \Theta$.

We will comment on those results after having discussed the lower semi-modular case.

5.3. Boolean and lattice independence in the lower semi-modular lattice $L^*(\Theta)$

Analogously, the extended matroidal independence relations associated with the lower semi-modular lattice $L^*(\Theta)$ read as

$$\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_1^* \quad \Leftrightarrow \quad \Theta_j \oplus \bigotimes_{i \neq j} \Theta_i \neq \Theta_j \quad \forall\, j = 1, ..., n, \tag{15}$$

$$\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_2^* \quad \Leftrightarrow \quad \Theta_j \oplus \bigotimes_{i=1}^{j-1} \Theta_i = \mathbf{0}_{\mathcal{F}} \quad \forall\, j = 2, ..., n, \tag{16}$$

$$\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_3^* \quad \Leftrightarrow \quad \Big|\bigotimes_{i=1}^{n} \Theta_i\Big| - 1 = \sum_{i=1}^{n} (|\Theta_i| - 1), \tag{17}$$

as $\Theta_i \wedge^* \Theta_j = \Theta_i \oplus \Theta_j$, $\Theta_i \vee^* \Theta_j = \Theta_i \otimes \Theta_j$, $h^*(\Theta_i) = |\Theta_i| - 1$, and $\mathbf{0} = \mathbf{0}_{\mathcal{F}}$.

The frames $\{\Theta_1, ..., \Theta_n\}$ are $\mathcal{I}_1^*$ iff none of them is a coarsening of the minimal refinement of all the others. In other words, there is no proper subset of $\{\Theta_1, ..., \Theta_n\}$ which has still $\Theta_1 \otimes ... \otimes \Theta_n$ as common refinement. $\Theta_1, ..., \Theta_n$ are $\mathcal{I}_2^*$ iff $\forall j > 1$ Θ_j does not have a non-trivial common coarsening with the minimal refinement of its predecessors.
Again, the third form $\mathcal{I}_3^*$ of extended matroidal independence relation is the most interesting. It has indeed a very interesting semantics in terms of probability spaces: As the dimension of the polytope of probability measures definable on a domain of size k is $k-1$, $\Theta_1, ..., \Theta_n$ are $\mathcal{I}_3^*$ iff the dimension of the probability polytope for the minimal refinement is the sum of the dimensions of the polytopes associated with the individual frames.

$$\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_3^* \equiv dim\mathcal{P}_{\bigotimes_{i=1}^n \Theta_i} = \sum_i \dim \mathcal{P}_{\Theta_i}. \tag{18}$$

Accordingly, as $\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_3$ is equivalent to

$$\begin{aligned} |\Theta| - \left| \bigoplus_{i=1}^n \Theta_i \right| &= \sum_{i=1}^n (|\Theta| - |\Theta_i|) \equiv \dim \mathcal{P}_\Theta - \dim \mathcal{P}_{\bigoplus_{i=1}^n \Theta_i} \\ &= \sum_i (\dim \mathcal{P}_\Theta - \dim \mathcal{P}_{\Theta_i}), \end{aligned}$$

its dual relation $\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_3^*$ can also be interpreted by saying that the difference of the dimensions of the probability simplices on Θ and $\bigoplus_{i=1}^n \Theta_i$ is the sum of the individual differences.
It is interesting to point out the following analogy between independence of frames and $\mathcal{I}_3$. While condition (4) for $\mathcal{IF}$

$$\Theta_1 \otimes \cdots \otimes \Theta_n = \Theta_1 \times \cdots \times \Theta_n$$

says that the minimal refinement is the Cartesian product of the individual frames, Equation (18) for $\mathcal{I}_3^*$ states that *the probability simplex* of the minimal refinement is a Cartesian product of the individual ones. We will consider their relationship in more detail in the last part of the paper.

5.3.1. *General case*
In this case it is easier to describe first the general framework, and later prove stronger statements holding in specific situations.

Theorem 7. *If* $\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF}$ *and* $\Theta_j \neq \mathbf{0}_\mathcal{F}$ $\forall j$, *then* $\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_1^*$.

Proof. Let us suppose that $\{\Theta_1, ..., \Theta_n\}$ are $\mathcal{IF}$ but not $\mathcal{I}_1^*$, i.e. $\exists j : \Theta_j$ coarsening of $\bigotimes_{i \neq j} \Theta_i$. We need to prove that $\exists A_1 \subset \Theta_1, ..., A_n \subset \Theta_n$ s.t.

$$\rho_1(A_1) \cap \cdots \cap \rho_n(A_n) = \emptyset,$$

where ρ_i denotes the refining from Θ_i to $\Theta_1 \otimes \cdots \otimes \Theta_n$.
Since Θ_j is a coarsening of $\bigotimes_{i \neq j} \Theta_i$ then there exists a partition Π_j of $\bigotimes_{i \neq j} \Theta_i$ associated with Θ_j, and a refining ρ from Θ_j to $\bigotimes_{i \neq j} \Theta_i$.
As $\{\Theta_i, i \neq j\}$ are $\mathcal{IF}$, for all $\theta \in \bigotimes_{i \neq j} \Theta_i$ there exist $\theta_i \in \Theta_i$, $i \neq j$ s.t.

$$\{\theta\} = \bigcap_{i \neq j} \rho'_i(\Theta_i),$$

where ρ'_i is the refining to $\bigotimes_{i \neq j} \Theta_i$. Now, θ belongs to a certain element A of the partition Π_j. By hypothesis ($\Theta_j \neq \mathbf{0}_{\mathcal{F}}\ \forall j$) Π_j contains at least two elements. But then we can choose $\theta_j = \rho^{-1}(B)$ with B another element of Π_j. In that case we obviously get

$$\rho_j(\theta_j) \cap \bigcap_{i \neq j} \rho_i(\theta_i) = \emptyset,$$

which implies that $\{\Theta_i, i = 1, ..., n\} \in \neg\mathcal{IF}$ against the hypothesis. □

Does $\mathcal{IF}$ imply $\mathcal{I}_1^*$ even when $\exists \Theta_i = \mathbf{0}_{\mathcal{F}}$? The answer is negative. $\{\Theta_1, ..., \Theta_n\} \in \neg\mathcal{I}_1^*$ means that $\exists i$ s.t. Θ_j is a coarsening of $\bigotimes_{i \neq j} \Theta_i$. But if $\Theta_i = \mathbf{0}_{\mathcal{F}}$ then Θ_i *is* a coarsening of $\bigotimes_{i \neq j} \Theta_i$.
The reverse implication does not hold: $\mathcal{IF}$ and $\mathcal{I}_1$ are distinct.

Theorem 8. *$\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_1^* \not\vdash \{\Theta_1, ..., \Theta_n\} \in \mathcal{IF}$.*

Proof. We need a simple counterexample. Consider two frames Θ_1 and Θ_2 in which Θ_1 is not a coarsening of Θ_2 (Θ_1, Θ_2 are $\mathcal{I}_1^*$). Then $\Theta_1, \Theta_2 \neq \Theta_1 \otimes \Theta_2$ but it easy to find an example (see Figure 5.3.1) in which Θ_1, Θ_2 are not $\mathcal{IF}$. □

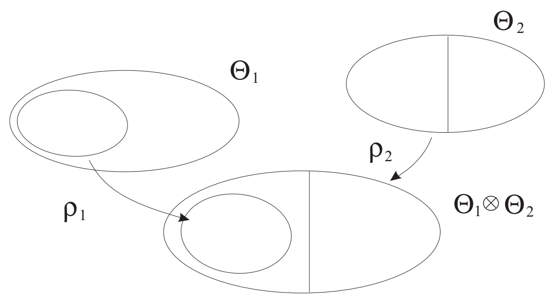

Figure 10. A counterexample to $\mathcal{I}_1^* \vdash \mathcal{IF}$.

Besides, like in the upper semi-modular case, $\mathcal{I}_2^*$ does not imply $\mathcal{I}_1^*$.

Theorem 9. *$\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_2^* \not\vdash \{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_1^*$.*

Proof. Figure 5.3.1 shows a counterexample to the conjecture $\mathcal{I}_2^* \vdash \mathcal{I}_1^*$. Given $\Theta_1 \otimes \cdots \otimes \Theta_{j-1}$ and Θ_j, one possible choice of Θ_{j+1} s.t. $\Theta_1, ..., \Theta_{j+1}$ are $\mathcal{I}_2^*$ but not $\mathcal{I}_1^*$ is shown. □

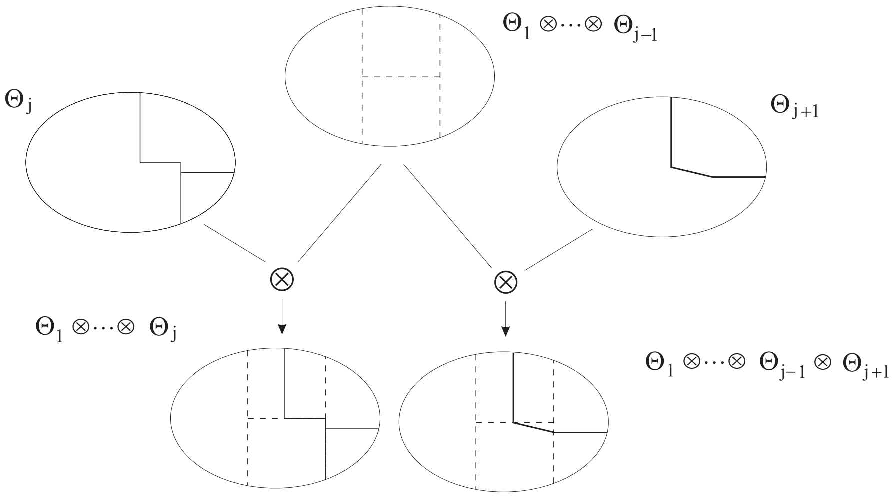

Figure 11. A counterexample to $\mathcal{I}_2^* \vdash \mathcal{I}_1^*$.

$\mathcal{IF}$ is a stronger condition than $\mathcal{I}_2^*$ also.

Theorem 10. $\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF} \vdash \{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_2^*$.

Proof. We first need to show that $\{\Theta_1, ..., \Theta_n\}$ are $\mathcal{IF}$ iff $\forall j = 1, ..., n$ the pair $\{\Theta_j, \otimes_{i\neq j}\Theta_i\}$ is $\mathcal{IF}$. As a matter of fact (4) can be written as

$$\Theta_j \otimes \bigotimes_{i\neq j} \Theta_i = \Theta_j \times (\times_{i\neq j}\Theta_i) \equiv \Big\{\Theta_j, \bigotimes_{i\neq j} \Theta_i\Big\} \in \mathcal{IF}.$$

But then by Lemma 1 we get as desired. □

It follows from Theorems 7 and 10 that, unless some frame is unitary,

Corollary 3. $\{\Theta_1, ..., \Theta_n\} \in \mathcal{IF} \vdash \{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_1^* \wedge \mathcal{I}_2^*$.

i.e. independence of frames is a more demanding requirement than both the first two forms of lattice-theoretic independence. The converse is however false. Think of a pair of frames ($n = 2$), for which

$$\Theta_1 \oplus \Theta_2 \neq \Theta_1, \Theta_2 \ (\{\Theta_1, \Theta_2\} \in \mathcal{I}_1^*), \qquad \Theta_1 \oplus \Theta_2 = \mathbf{0}_{\mathcal{F}} \ (\{\Theta_1, \Theta_2\} \in \mathcal{I}_2^*).$$

Such conditions are met for instance in the counterexample of Figure 5.3.1, in which the two frames are not $\mathcal{IF}$.

5.3.2. *Pairs of frames*
We can indeed maintain something stronger when considering only pairs of frames. For $n = 2$ $\{\Theta_1, \Theta_2\} \in \mathcal{I}_1^*$, $\{\Theta_1, \Theta_2\} \in \mathcal{I}_2^*$, $\{\Theta_1, \Theta_2\} \in \mathcal{I}_3^*$ read respectively as

$$\Theta_1 \oplus \Theta_2 \neq \Theta_1, \Theta_2, \quad \Theta_1 \oplus \Theta_2 = \mathbf{0}_{\mathcal{F}}, \quad |\Theta_1 \otimes \Theta_2| = |\Theta_1| + |\Theta_2| - 1. \tag{19}$$

Unlike the general case, for pairs of frames $\mathcal{I}_2^*$ does imply $\mathcal{I}_1^*$.

Theorem 11. *If* $\Theta_1, \Theta_2 \neq \mathbf{0}_{\mathcal{F}}$ *then* $\{\Theta_1, \Theta_2\} \in \mathcal{I}_2^*$ *implies* $\{\Theta_1, \Theta_2\} \in \mathcal{I}_1^*$ *.*

Proof. Obvious by Equation (19). □

If one of the frames is unitary, all independence conditions hold but $\mathcal{I}_1^*$.

Theorem 12. *If* $\exists \Theta_j = \mathbf{0}_{\mathcal{F}}$ $j \in \{1, 2\}$ *then* $\{\Theta_1, \Theta_2\} \in \mathcal{I}_2^*$*,* $\mathcal{I}_3^*$*,* $\mathcal{IF}$*,* $\neg\mathcal{I}_1^*$*.*

Proof. If for instance $\Theta_2 = \mathbf{0}_{\mathcal{F}}$ then by (19) $\Theta_1 \oplus \mathbf{0}_{\mathcal{F}} = \mathbf{0}_{\mathcal{F}} = \Theta_2$ and $\{\Theta_1, \Theta_2\}$ are not $\mathcal{I}_1^*$ while they are $\mathcal{I}_2^*$. As $|\Theta_1 \otimes \Theta_2| = |\mathbf{0}_{\mathcal{F}}| \cdot |\Theta_1| = |\Theta_1|$, $|\Theta_2| = 1$ they are $\mathcal{I}_3^*$ again by (19). Finally, their are $\mathcal{IF}$ as $|\Theta_1 \otimes \Theta_2| = |\Theta_1| = 1 \cdot |\Theta_1| = |\Theta_2| \cdot |\Theta_1|$ (according to Equation (4)). □

Can we have non-unitary pairs of frames which are both $\mathcal{I}_3^*$ and $\mathcal{I}_1^*$? Figure 2 provides such an example: as $|\Theta_1 \otimes \Theta_2| = 4 = |\Theta_1| + |\Theta_2| - 1 = 3 + 2 - 1$ the two frames are $\mathcal{I}_3^*$, none of them is a refinement of the other, and they are both non-unitary.

5.3.3. *Atomic case*
The atoms of the lattice $L^*(\Theta)$ are nothing but the binary partitions of Θ, i.e. $\{\Omega \in L^*(\Theta) : |\Omega| = 2\}$. For them, both $\mathcal{I}_1, \mathcal{I}_2$ are trivial.

Theorem 13. *If* $\{\Theta_1, ..., \Theta_n\} \in A^*$ *then* $\{\Theta_1, ..., \Theta_n\} \in \mathcal{I}_1^* \wedge \mathcal{I}_2^*$*.*

Proof. If $\Theta_j \in A^*$ $\forall j = 1, ..., n$ then $\Theta_j \oplus \bigotimes_{i \neq j} \Theta_i = \mathbf{0}_{\mathcal{F}} \neq \Theta_j$ $\forall j$ and $\{\Theta_1, ..., \Theta_n\}$ are $\mathcal{I}_1^*$. But then by Definition 16 $\{\Theta_1, ..., \Theta_n\}$ are also $\mathcal{I}_2^*$. □

5.4. Comments

Figure 5.4 illustrates what we have learned about the relations between independence of frames and the various extensions of matroidal independence to semi-modular lattices, in both the upper (left) and lower (right) semi-modular lattice of frames. Only the general case of

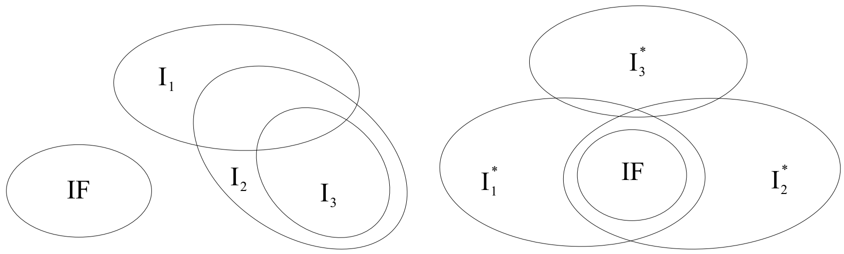

Figure 12. Left: Relations between independence of frames $\mathcal{IF}$ and all the different extended forms of matroidal independence on the upper semi-modular lattice $L(\Theta)$. Right: Relations on the lower semi-modular lattice $L^*(\Theta)$.

a collection of more than two non-atomic frames is shown for sake of simplicity: special cases ($\Theta_i = \mathbf{0}_\mathcal{F}$ for $L^*(\Theta)$, $\Theta_i = \Theta$ for $L(\Theta)$) are also neglected.

In the upper semi-modular case, minding the special case in which one of the frames is Θ itself, independence of frames $\mathcal{IF}$ is mutually exclusive with all lattice-theoretic relations $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ (Theorems 5, 6 and Corollary 1) unless we consider two non-atomic frames, for which $\mathcal{IF} \vdash \mathcal{I}_1$ (Theorem 4). In fact they are *the negation* of each other in the case of atoms of $L(\Theta)$ (frames of size $n-1$), when $\mathcal{I} = \mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3$ is trivially true for all frames, while $\mathcal{IF}$ is never met (Theorem 3). The exact relation between $\mathcal{I}_1$ and $\mathcal{I}_2$, $\mathcal{I}_3$ is not yet understood, but we know that the latter imply the former when dealing with pairs.
In the lower semi-modular case $\mathcal{IF}$ is *a stronger condition* than both $\mathcal{I}_1^*$ and $\mathcal{I}_2^*$ (Theorems 7, 10) which are indeed trivial for binary partitions of Θ (Theorem 13). On the other side, as we will prove in the following, and notwithstanding the analogy coming from Equation (18), $\mathcal{IF}$ is mutually exclusive with the third independence relation even in its lower semi-modular incarnation.

Some common features emerge: the first two forms of lattice independence are always trivially met by atoms of the related lattice. More, independence of frames and the third form of lattice independence are mutually exclusive in either case.
The lower semi-modular case is clearly more interesting, though. On $L(\Theta)$ independence of frames and lattice-theoretic independence are basically unrelated (see Figure 5.4-left). Their lower-semi-modular counterparts, instead, have meaningful links with $\mathcal{IF}$ even though distinct from it. The knowledge of which collections of frames are $\mathcal{I}_1^*$, $\mathcal{I}_2^*$ and $\mathcal{I}_3^*$ tells us much about $\mathcal{IF}$ frames, as collections of Boolean independent frames are in

$$\mathcal{I}_1^* \cap \mathcal{I}_2^* \cap \neg\mathcal{I}_3^*.$$

We know that $\mathcal{IF}$ is *strictly* included in $\mathcal{I}_1^* \cap \mathcal{I}_2^*$ (Section 5.3.1), but the possibility that independence of frames may indeed coincide with $\mathcal{I}_1^* \cap \mathcal{I}_2^* \cap \neg\mathcal{I}_3^*$ is still to be explored.

6. Independence of frames opposed to matroidal independence

So far we tried to reduce independence of frames to some extended form of matroidal independence. However, $(L(\Theta), \mathcal{IF})$ is not a matroid, and $\mathcal{IF}$ cannot be explained as a criptomorphic form of any of the extended matroidal independence relations (Section 5.4), even though they are indeed strictly related. We close this paper by showing that independence of frames is in fact *in opposition* to matroidal independence.
According to Proposition 4, all collections of atoms are $\mathcal{I} = \mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3$ in the lattice $L(\Theta)$. More interesting is the case of the lattice $L^*(\Theta)$, in which the order relation is $\Theta_1 \leq^* \Theta_2$ iff Θ_1 is a coarsening of Θ_2.

6.1. The case of independent binary frames

Let us consider the partition lattice associated with a frame $\Theta = \{1, 2, 3, 4\}$ of cardinality 4. There cannot be $\mathcal{IF}$ collections of three or more frames, as in that case the size of their minimal refinement should be (by Equation 4) $2 \times 2 \times 2 = 8$ for them to be $\mathcal{IF}$. Let us then focus on all pairs of binary partitions (the atoms of $L^*(\Theta)$).
By looking at Figure 4.4.1 we can notice that the only $\mathcal{IF}$ pairs are formed by two of the following binary partitions

$$1, 2/3, 4 \qquad 1, 3/2, 4 \qquad 1, 4/2, 3.$$

(dark nodes in the diagram) as their maximal coarsening is $\Theta = \{1, 2, 3, 4\}$ and has cardinality $2 \times 2 = 4$. But it is easy to see that all the other binary pairs are $\mathcal{I}$. We can prove that in the general case.
Let us consider the form $\mathcal{I}_3$ of matroidal independence. As in $L^*(\Theta)$ the sup is the minimal refinement, (17) reads as

$$h(\Theta_1 \otimes \cdots \otimes \Theta_n) = \sum_i h(\Theta_i) \equiv |\Theta_1 \otimes \cdots \otimes \Theta_n| - 1 = \sum_i (|\Theta_i| - 1) \tag{20}$$

which for $n = 2$ reads as $|\Theta_1 \otimes \Theta_2| = |\Theta_1| + |\Theta_2| - 1$. For pairs of binary frames $\mathcal{IF}$ can be written as $|\Theta_1 \otimes \Theta_2| = |\Theta_1| \cdot |\Theta_2| = 2 \cdot 2 = 4$. For all the binary pairs which are not $\mathcal{IF}$ we have $|\Theta_1 \otimes \Theta_2| = 3$ (see Figure 4.4.1 again) and as there are no other possible values for $|\Theta_1 \otimes \Theta_2|$, $\mathcal{I} = \neg\mathcal{IF}$. In conclusion,

Theorem 14. *Pairs of binary partitions of a frame Θ (atoms of the lattice $L^*(\Theta)$) are independent as frames ($\mathcal{IF}$) if and only if they are not independent as elements of a matroid ($\mathcal{I}$).*

6.2. Mutual exclusivity of $\mathcal{I}_3$ and $\mathcal{IF}$

We can in fact prove a stronger, more general statement. Recall that $\mathcal{I}_3$ is the generalization of $\mathcal{I}$ to arbitrary elements of $L(\Theta)$.

Theorem 15. *If a collection $\{\Theta_1, ..., \Theta_n\}$ of compatible frames is $\mathcal{IF}$ then it is not $\mathcal{I}_3$, unless $n = 2$ and one of the frames is the trivial partition.*

Proof. According to Equation (4), $\{\Theta_1, ..., \Theta_n\}$ are $\mathcal{IF}$ iff $|\otimes \Theta_i| = \prod_i |\Theta_i|$, while according to (17) they are $\mathcal{I}$ iff $|\Theta_1 \otimes \cdots \otimes \Theta_n| - 1 = \sum_i(|\Theta_i| - 1)$. Those conditions are both met iff

$$\sum_i |\Theta_i| - \prod_i |\Theta_i| = n - 1$$

which happens only if $n = 2$ and either $\Theta_1 = \mathbf{0}_{\mathcal{F}}$ or $\Theta_1 = \mathbf{0}_{\mathcal{F}}$. □

Instead of being algebraically related notions, independence of frames and matroidicity work against each other. As independence of frames derives from independence of Boolean subalgebras of a Boolean algebra (Sikorski, 1964), this is likely to have interesting wider implications on the relationship between independence in those two fields of mathematics.

7. Conclusions

In this paper we gave a rather exhaustive description of families of compatible frames in terms of the algebraic structures they form: Boolean sub-algebras, upper, and lower semi-modular lattices. Each of those comes with a characteristic form of independence. We compared them with Shafer's notion of independence of frames, in a pursuit for an algebraic interpretation of independence in the theory of evidence.
Even though $\mathcal{IF}$ cannot be explained in terms of classical matroidal independence, it possesses interesting relations with its extended forms on semi-modular lattices. It turns out that independence of frames is actually opposed to matroidal independence, a rather surprising result. Even though this can be seen as a negative result in the original perspective of finding an algebraic solution to the problem of merging conflicting belief function on non-independent frames, we now understand

much better where independence of frames stands from an algebraic point of view. New lines of research remain open, for instance in what concerns an explanation of independence of frames as independence of flats in a geometric lattice (Cuzzolin, 2008a). We believe the prosecution of this study could in the future shed some more light on both the nature of independence of sources in the theory of subjective probability, and the relationship between matroidal and Boolean independence in discrete mathematics, pointing out the necessity of a more general, comprehensive definition of this very important notion.

Appendix

Families of compatible frames

Definition 7. *A non-empty collection of finite non-empty sets $\mathcal{F}$ is a* family of compatible frames of discernment *((*Shafer, 1976*), pages 121-125) with refinings $\mathcal{R}$, where $\mathcal{R}$ is a non-empty collection of refinings between couples of frames in $\mathcal{F}$, if $\mathcal{F}$ and $\mathcal{R}$ meet the following requirements:*

1. composition of refinings*: if $\rho_1 : 2^{\Theta_1} \rightarrow 2^{\Theta_2}$ and $\rho_2 : 2^{\Theta_2} \rightarrow 2^{\Theta_3}$ are in $\mathcal{R}$, then $\rho_1 \circ \rho_2 \in \mathcal{R}$.*

2. identity of coarsenings*: if $\rho_1 : 2^{\Theta_1} \rightarrow 2^{\Theta'}$ and $\rho_2 : 2^{\Theta_2} \rightarrow 2^{\Theta'}$ are in $\mathcal{R}$ and $\forall\ \theta_1 \in \Theta_1\ \exists \theta_2 \in \Theta_2$ such that $\rho_1(\{\theta_1\}) = \rho_2(\{\theta_2\})$, then $\Theta_1 = \Theta_2$ and $\rho_1 = \rho_2$.*

3. identity of refinings*: if $\rho_1 : 2^{\Theta} \rightarrow 2^{\Theta'}$ and $\rho_2 : 2^{\Theta} \rightarrow 2^{\Theta'}$ are in $\mathcal{R}$, then $\rho_1 = \rho_2$.*

4. existence of coarsenings*: if $\Theta \in \mathcal{F}$ and $A_1, ..., A_n$ is a disjoint partition of Θ, then there is a coarsening Θ' of Θ in $\mathcal{F}$ corresponding to this partition, i.e. $\forall A_i$ there exists an element of Θ' whose image under the appropriate refining is A_i.*

5. existence of refinings*: if $\theta \in \Theta \in \mathcal{F}$ and $n \in \mathbb{N}$ then there exists a refining $\rho : 2^{\Theta} \rightarrow 2^{\Theta'}$ in $\mathcal{R}$ and $\Theta' \in \mathcal{F}$ such that $\rho(\{\theta\})$ has n elements.*

6. existence of common refinements*: every pair of elements in $\mathcal{F}$ has a common refinement in $\mathcal{F}$.*

Independence of frames as independence of Boolean sub-algebras

Definition 8. *A* Boolean algebra *is a non-empty set $\mathcal{B}$ provided with three internal operations*

$$\begin{array}{rcl} \cap: \mathcal{B}\times\mathcal{B} & \longrightarrow & \mathcal{B} \\ A,B & \mapsto & A\cap B \end{array} \qquad \begin{array}{rcl} \cup: \mathcal{B}\times\mathcal{B} & \longrightarrow & \mathcal{B} \\ A,B & \mapsto & A\cup B \end{array} \qquad \begin{array}{rcl} \neg: \mathcal{B} & \longrightarrow & \mathcal{B} \\ A & \mapsto & \neg A \end{array}$$

called respectively meet*,* join *and* complement*, characterized by the following properties:*

$$\begin{array}{ccr} A\cup B = B\cup A, & A\cap B = B\cap A, & (A_1) \\ A\cup(B\cup C) = (A\cup B)\cup C, & A\cap(B\cap C) = (A\cap B)\cap C, & (A_2) \\ (A\cap B)\cup B = B, & (A\cup B)\cap B = B, & (A_3) \\ A\cap(B\cup C) = (A\cap B)\cup(A\cap C), & A\cup(B\cap C) = (A\cup B)\cap(A\cup C), & (A_4) \\ (A\cap\neg A)\cup B = B, & (A\cup\neg A)\cap B = B. & (A_5) \end{array}$$

For instance, the power set 2^Θ of a set Θ is a Boolean algebra. $X = 2^{\Theta'}$ with Θ' a disjoint partition of Θ is then a Boolean *sub-algebra* (Sikorski, 1964) of 2^Θ. A collection of compatible frames $\Theta_1, ..., \Theta_n$ corresponds then to a collection of Boolean sub-algebras $2^{\Theta_1}, ..., 2^{\Theta_n}$ of the power set $2^{\Theta_1\otimes\cdots\otimes\Theta_n}$ of their minimal refinement. Now, a collection of Boolean sub-algebras $X_1, ..., X_n$ is *independent* ($\mathcal{IB}$) if

$$\cap A_i \neq \wedge \qquad (21)$$

$\forall\ A_i \in X_i$, where $\wedge \doteq \cap\mathcal{B}$ is the initial element of the Boolean algebra. For a collection of compatible frames (21) is expressed as (3).

References

Aigner, M.: 1979, *Combinatorial Theory.* New York: Classics in Mathematics, Springer.

Beutelspacher, A. and U. Rosenbaum: 1998, *Projective geometry.* Cambridge: Cambridge University Press.

Birkhoff, G.: 1935, 'Abstract linear dependence and lattices'. *American Journal of Mathematics* **57**, 800–804.

Campos, F. and F. de Souza: 2005, 'Extending Dempster-Shafer theory to overcome counter intuitive results'. In: *Proceedings of IEEE NLP-KE '05*, Vol. 3. pp. 729–734.

Carlson, J. and R. Murphy: 2005, 'Use of Dempster-Shafer Conflict Metric to Adapt Sensor Allocation to Unknown Environments'. Technical report, Safety Security Rescue Research Center, University of South Florida.

Cattaneo, M. E. G. V.: 2003, 'Combining Belief Functions Issued from Dependent Sources.'. In: *ISIPTA*. pp. 133–147.

Cuzzolin, F.: 2005a, 'Algebraic structure of the families of compatible frames of discernment'. *AMAI* **45(1-2)**, 241–274.
Cuzzolin, F.: 2005b, 'Evidential modeling for pose estimation'. In: *Proc. of ISIPTA*.
Cuzzolin, F.: 2008b, 'On the structure of simplex of inner Bayesian approximations'. In: *Proceedings of the European Conference on Logics in Artificial Intelligence (JELIA'08), Dresden, Germany*.
Cuzzolin, F.: Fort Lauderdale, January 2-4 2008a, 'Boolean and matroidal independence in uncertainty theory'. In: *Proc. of the 10th International Symposium on Mathematics and Artificial Intelligence ISAIM'08*.
Dempster, A.: 1968, 'Upper and lower probabilities generated by a random closed interval'. *Annals of Mathematical Statistics* **39**, 957–966.
Dempster, A.: 1969, 'Upper and lower probabilities inferences for families of hypothesis with monotone density ratios'. *Annals of Mathematical Statistics* **40**, 953–969.
Dempster, A. P.: 1967, 'Upper and lower probability inferences based on a sample from a finite univariate population'. *Biometrika* **54**, 515–528.
Deutsch-McLeish, M.: Paris, France, 2-6 July 1990, 'A study of probabilities and belief functions under conflicting evidence: comparisons and new method'. In: *Proceedings of IPMU'90*. pp. 41–49.
Dubois, D. and H. Prade: 1992, 'On the combination of evidence in various mathematical frameworks'. In: J. flamm and T. Luisi (eds.): *Reliability Data Collection and Analysis*. pp. 213–241.
Harary, F. and W. T. Tutte: 1969, 'Matroids versus graphs'. In: *The many facets of graph theory, Lecture Notes in Math., Vol. 110*. Springer-Verlag, Berlin, pp. 155–170.
Josang, A., M. Daniel, and P. Vannoorenberghe: 2003, 'Strategies for Combining Conflicting Dogmatic Beliefs'. In: *Proceedings of Fusion 2003*, Vol. 2. pp. 1133–1140.
Kohlas, J. and P.-A. Monney: 1995, *A Mathematical Theory of Hints - An Approach to the Dempster-Shafer Theory of Evidence*. Lecture Notes in Economics and Mathematical Systems, Springer-Verlag.
Lane, S. M.: 1938, 'A lattice formulation for transcendence degrees and p-bases'. *Duke Math. J.* **4**, 455–468.
Lefevre, E., O. Colot, and P. Vannoorenberghe: 2002, 'Belief functions combination and conflict management'. *Information Fusion Journal* **3**(2), 149–162.
Liu, W.: 2006, 'Analyzing the degree of conflict among belief functions'. *Artif. Intell.* **170**(11), 909–924.
Murphy, C.: 2000, 'Combining belief functions when evidence conflicts'. *Decision Support Systems* **29**, 1–9.
Nguyen, H.: 1978, 'On Random Sets and Belief Functions'. *J. Mathematical Analysis and Applications* **65**, 531–542.
Oxley, J. G.: 1992, *Matroid theory*. Great Clarendon Street, Oxford, UK: Oxford University Press.
Sentz, K. and S. Ferson: April 2002, 'Combination of Evidence in Dempster-Shafer Theory'. Technical report, SANDIA Tech. Report, SAND2002-0835.
Shafer, G.: 1976, *A Mathematical Theory of Evidence*. Princeton University Press.
Shafer, G.: 1990, 'Perspectives on the theory and practice of belief functions'. *International Journal of Approximate Reasoning* **4**, 323–362.
Shafer, G., P. Shenoy, and K. Mellouli: 1987, 'Propagating belief functions in qualitative Markov trees'. *IJAR* **1**, (4), 349–400.
Sikorski, R.: 1964, *Boolean algebras*. Springer Verlag.

Smets, P.: 1990, 'The Combination of Evidence in the Transferable Belief Models'. *IEEE Transactions on PAMI* **12**, 447–458.

Smets, P.: Guangzhou, China, 1987, 'Upper and lower probability functions versus belief functions'. In: *Proceedings of the International Symposium on Fuzzy Systems and Knowledge Engineering*. pp. 17–21.

Stern, M.: 1999, *Semimodular lattices.* Cambridge University Press.

Szasz, G.: 1963, *Introduction to lattice theory.* New York: Academic Press.

Teichmuller, O.: 1936, 'p-Algebren'. *Deutsche Math.* **1**, 362–388.

van der Waerden, B. L.: 1937, *Moderne Algebra, Vol. 1.* Berlin: Springer-Verlag.

Whitney, H.: 1935, 'On the abstract properties of linear dependence'. *American Journal of Mathematics* **57**, 509–533.

Wierman, M.: 2001, 'Measuring conflict in evidence theory'. In: *Proceedings of the Joint 9th IFSA World Congress, Vancouver, BC, Canada*, Vol. 3. pp. 1741–1745.

Yager, R.: 1987, 'On the Dempster-Shafer framework and new combination rules'. *Information Sciences* **41**, 93–138.

Zadeh, L.: 1986, 'A simple view of the Dempster-Shafer theory of evidence and its implication for the rule of combination'. *AI Magazine* **7**(2), 85–90.