On the fiber bundle structure of the space of belief functions

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Abstract

The study of finite non-additive measures or "belief functions" has been recently posed in connection with combinatorics and convex geometry. As a matter of fact, as belief functions are completely specified by the associated belief values on the events of the frame on which they are defined, they can be represented as points of a Cartesian space. The space of all belief functions $\mathcal{B}$ or "belief space" is a simplex whose vertices are BF focused on single events. In this paper we present an alternative description of the space of belief functions in terms of differential geometric notions. The belief space possesses indeed a recursive bundle structure inherently related to the mass assignment mechanism, in which basic probability is recursively assigned to events of increasing size. A formal proof of the decomposition of $\mathcal{B}$ together with a characterization of bases and fibers as simplices are provided.

I. Introduction

The *theory of evidence* [19] was introduced in the late Seventies by Glenn Shafer as a way of representing subjective probabilities or degrees of belief, starting from a sequence of seminal works [10] by Arthur Dempster. In this formalism knowledge states are represented by *belief functions* (BFs) rather than classical probability distributions. The notion of belief function was originally introduced by A. Dempster as a simple consequence of the application of probability theory to multiple domains linked by a multi-valued map [10]. A belief function is in fact the mathematical object induced on the codomain of a multivariate mapping by a probability living on its domain. However, equivalent alternative interpretations of BFs can be given in terms of random sets [17], [14], compatibility relations, inner measures [18], [12].

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As mathematical objects, though, belief functions can be assimilated to *sum functions*, i.e., functions on the power set $2^\Theta = \{A \subseteq \Theta\}$ of a finite domain Θ of the form

$$b(A) = \sum_{\emptyset \subsetneq B \subseteq A} m(B)$$

induced by another function $m : 2^\Theta \rightarrow [0, 1]$ called *basic probability assignment*. This obviously allows to establish interesting connections between probability theory and combinatorics. In particular, the nature of BFs as peculiar sum functions reveals a deep connection between theory of evidence and convex geometry. We have recently introduced a geometric approach of the theory of evidence [4] in which belief functions are seen as points of a Cartesian space. Indeed, a belief function $b : 2^\Theta \rightarrow [0, 1]$ is completely specified by its $N - 2$ ($N = 2^{|\Theta|}$) belief values $\{b(A), \forall \emptyset \subsetneq A \subsetneq \Theta\}$ and can therefore be thought of as a vector $v = [v_A = b(A), \emptyset \subsetneq A \subsetneq \Theta]'$ of $\mathbb{R}^{N-2}$. The collection $\mathcal{B}$ of all the points of $\mathbb{R}^{N-2}$ which correspond to a belief function is a simplex, whose vertices are BFs focusing on a single event A ($m(A) = 1$, $m(B) = 0 \ \forall B \neq A$). The basic probability assignment m associated with a BF b can then be geometrically interpreted as the set of simplicial coordinates of b in $\mathcal{B}$.

The geometric approach to the theory of belief functions is due to the author [4], and has been extended to the study of the geometry of possibility measures [6]. In particular, the problem of transforming a belief function into a probability measure has been studied, and two approximations derived from geometric arguments introduced [3]. The geometric behavior of Dempster's rule of combination [10] has also been thoroughly investigated, and the properties of the approximations inherently associated with Dempster's sum studied in this geometric framework [7], [5]. Nevertheless, the study of the relationship between belief functions and probabilities has been posed in a geometric setup by other authors [13], [2]. In this context, the problem of approximating a belief function with a probability has been studied by M. Daniel [9]. ADD

As we show in this paper, we can also think of the mass $m(A)$ given to each event A as *recursively* assigned to subsets of increasing size. Geometrically, this translates as a recursive decomposition of the space of belief functions which can be formally described through the differential-geometric notion of *fiber bundle* [11].

A fiber bundle is a generalization of the familiar idea of Cartesian product, in which each point of the (total) space analyzed can be smoothly projected onto a *base space*, defining a number of *fibers* of points which project onto the same element of the base. In our case, as we will see in the following, $\mathcal{B}$ can be decomposed $n = |\Theta|$ times into bases and fibers which are themselves simplices and possess natural interpretations in terms of degrees of belief. Each level $i = 1, .., n$ of this decomposition reflects nothing but the assignment of basic probabilities to size i events.

We will first (Section II) briefly recall the sum function interpretation of belief functions, and the simplicial form of the space of belief functions, to prove in Section III its recursive bundle structure. After giving an informal presentation of the way the b.p.a. mechanism induces a recursive decomposition of $\mathcal{B}$ we will analyze the simple case study of a ternary frame (III-A) to get an intuition on how to prove our conjecture on the bundle structure of the belief space in the general case, and give the formal definition of smooth fiber bundle (III-B). After noticing that points of $\mathbb{R}^{N-2}$ outside the belief space can be also seen as (normalized) sum functions (Section III-C), we will proceed to prove the recursive bundle structure of the space of all sum functions (Section III-D). As $\mathcal{B}$ is immersed in this Cartesian space it inherits a "pseudo" bundle structure (III-E) in which bases and fibers are no more vector spaces but simplices in their own right (Section III-F), and possess meanings in terms of i-additive belief functions.

II. The geometry of belief functions

The term "belief function" was coined by Glenn Shafer [19] to indicate a mathematical object designed to represent the available evidence in the framework of subjective probability. Mathematically speaking, however, Shafer's axiomatic definition of belief functions is only one of several alternatives [17], [18]. The most successful representation, though, assimilates belief functions to *sum functions* [1] through the notion of *basic probability assignment*.

Definition 1. *A* basic probability assignment *(b.p.a.) over a finite set (*frame of discernment *[19])* Θ *is a function* $m : 2^\Theta \to [0, 1]$ *on its power set* 2^Θ *such that:* $m(\emptyset) = 0$, $\sum_{A\subseteq\Theta} m(A) = 1$, $m(A) \geq 0 \ \forall A \subseteq \Theta$.

Subsets of Θ associated with non-zero values of m are called *focal elements*, and their union $\mathcal{C}$ *core*.

Definition 2. *The* belief function *(BF)* $b : 2^\Theta \to [0, 1]$ *associated with the basic probability assignment* m_b *on* Θ *is defined as:* $b(A) = \sum_{B\subseteq A} m_b(B)$.

A belief function b is therefore the sum function [1] associated with a basic probability assignment m_b on the partially ordered set $(2^\Theta, \subseteq)$, and can be viewed as the total probability induced by the generalized mass assignment m_b. Given a belief function b the corresponding basic probability assignment can be obtained by applying the *Möbius inversion lemma*[1]

$$m_b(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} b(B). \tag{1}$$

Finite probability measures on Θ are just special (*Bayesian*) BFs whose b.p.a. assigns non-zero mass to elements only: $m_b(A) = 0 \; \forall A : |A| > 1$, where $|A|$ denotes the cardinality of a subset A.

We will denote a Bayesian BF by $p : 2^\Theta \to [0, 1]$.

Simplicial geometry of belief functions. Now, consider a frame of discernment Θ and introduce in the Cartesian space $\mathbb{R}^{N-2}$, $N = 2^{|\Theta|}$ an orthonormal reference frame $\{X_A : \emptyset \subsetneq A \subsetneq \Theta\}$. Each vector $v = \sum_{\emptyset \subsetneq A \subsetneq \Theta} v_A X_A$ in $\mathbb{R}^{N-2}$ is therefore potentially a belief function, in which each component v_A measures the belief value of A: $v_A = b(A)$. We call *belief space* associated with Θ the set of points $\mathcal{B}_\Theta$ of $\mathbb{R}^{N-2}$ which correspond to a belief function. We will assume the domain Θ fixed, and denote the belief space simply by $\mathcal{B}$.

To determine which points of "are" belief functions we can exploit the Moebius inversion lemma (1), by computing the corresponding b.p.a. and checking the axioms m_b must obey. Let us denote by

$$b_A \doteq b \in \mathcal{B} \; s.t. \; m_{b_A}(A) = 1, \; m_{b_A}(B) = 0 \; \forall B \subseteq \Theta \; s.t. \; B \neq A \tag{2}$$

the *categorical* [20] belief function assigning all the mass to a single subset $A \subseteq \Theta$. It can be proved that [4]:

Proposition 1. *The belief space $\mathcal{B}$ coincides with the convex closure of all the categorical belief functions b_A, $\mathcal{B} = Cl(b_A, \emptyset \subsetneq A \subseteq \Theta)$, where Cl denotes the convex closure operator:*

$$Cl(b_1, ..., b_k) = \left\{ b \in \mathcal{B} : b = \alpha_1 b_1 + \cdots + \alpha_k b_k, \sum_i \alpha_i = 1, \; \alpha_i \geq 0 \; \forall i \right\}. \tag{3}$$

The convex closure of a collection of affinely independent [6] points is called a *simplex*. As the categorical BFs b_A are affinely independent, $\mathcal{B}$ is a simplex [4]. Moreover, each belief function

[1] See [1] for an explanation in terms of the theory of monotone functions over partially ordered sets.

$b \in \mathcal{B}$ can be written as a convex sum as:

$$b = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) b_A. \tag{4}$$

Geometrically the b.p.a. m_b of a BF b is the set of simplicial coordinates of b in the simplex $\mathcal{B}$. Since a probability measure is a belief function assigning non zero masses to singletons only, the set $\mathcal{P}$ of all the Bayesian belief functions is the simplex determined by all categorical BFs associated with singletons [2]: $\mathcal{P} = Cl(b_x, x \in \Theta)$.

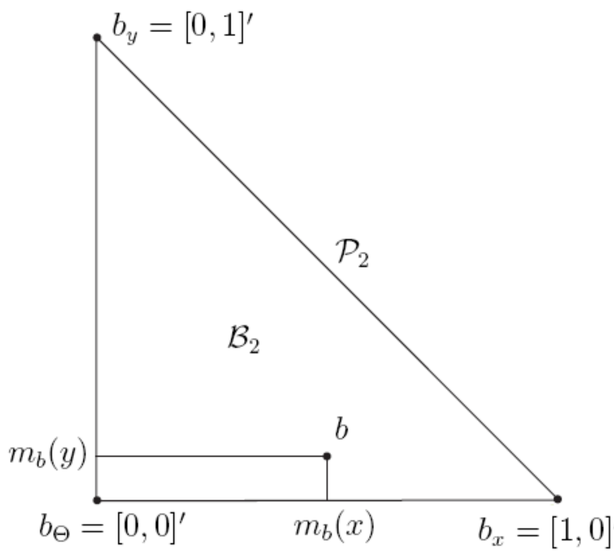

Fig. 1. The belief space $\mathcal{B}$ for a binary frame is a triangle in $\mathbb{R}^2$ whose vertices are the basis belief functions focused on $\{x\}, \{y\}$ and Θ, b_x, b_y, b_Θ respectively.

Binary example. As an example let us consider a frame of discernment containing only two elements, $\Theta_2 = \{x, y\}$. In this very simple case each BF $b : 2^{\Theta_2} \rightarrow [0,1]$ is completely determined by its belief values $b(x), b(y)$, as $b(\Theta) = 1$ and $b(\emptyset) = 0 \ \forall b$. We can therefore collect them in a vector of $\mathbb{R}^{N-2} = \mathbb{R}^2$ (since $N = 2^2 = 4$):

$$[b(x) = m_b(x), b(y) = m_b(y)]' \in \mathbb{R}^2. \tag{5}$$

[2]With a harmless abuse of notation we will denote the categorical belief function associated with a singleton x by b_x instead of $b_{\{x\}}$. Accordingly we will write $m_b(x), pl_b(x)$ instead of $m_b(\{x\}), pl_b(\{x\})$.

Since $m_b(x) \geq 0$, $m_b(y) \geq 0$, and $m_b(x) + m_b(y) \leq 1$ we can easily infer that the set $\mathcal{B}_2$ of all the possible belief functions on Θ_2 can be depicted as the triangle in the Cartesian plane of Figure 1, whose vertices are the points

$$b_\Theta = [0, 0]', \quad b_x = [1, 0]', \quad b_y = [0, 1]'$$

which correspond (through Equation (5)) respectively to the vacuous belief function b_Θ ($m_{b_\Theta}(\Theta) = 1$), the Bayesian BF b_x with $m_{b_x}(x) = 1$, and the Bayesian BF b_y with $m_{b_y}(y) = 1$.
The versors $X_x = [1, 0]', X_y = [0, 1]'$ form a reference frame $\{X_A : \emptyset \subsetneq A \subsetneq \Theta\}$ in the Cartesian plane. The Bayesian belief functions on Θ_2 obey the constraint $m_b(x) + m_b(y) = 1$, and are therefore located on the segment $\mathcal{P}_2$ joining $b_x = [1, 0]'$ and $b_y = [0, 1]'$.

III. Bundle structure of the belief space

We have seen that the sum function interpretation of belief functions is responsible for the simplicial structure of the associated space. The same basic probability assignment mechanism, though, induces an additional structure on $\mathcal{B}$. If we imagine to assign basic probability recursively to events of increasing size we obtain a description of the space of belief functions in terms of *fiber bundles*, a well known concept of differential geometry.

We will start by studying the simplest case in which this bundle decomposition of $\mathcal{B}$ arises, i.e., a frame of discernment $\Theta_3 = \{x, y, z\}$ composed by just three elements. Then, after reinterpreting this case study at the light of the formal definition of smooth fiber bundle (Section III-B), we will proceed to give a formal proof of the recursive quasi-bundle structure of $\mathcal{B}$. This will be done in two steps. We will prove in the first place (III-D) the recursive fiber bundle decomposition of the space of *normalized sum functions* (Section III-C), i.e., objects similar to BF which however do not necessarily meet the non-negativity constraint and correspond to arbitrary points of $\mathcal{S} = \mathbb{R}^{N-2}$. Then (III-E) we will proceed to show how the non-negativity and normalization constraints imposed on basic probability assignments act on the structure of $\mathcal{S}$, inducing a "pseudo" bundle structure in the simplicial belief space $\mathcal{B}$ in which basis and fibers are themselves simplices.

A. A case study: the ternary case

Let us then first consider the structure of the belief space for a frame of cardinality $n = 3$: $\Theta = \{x, y, z\}$, according to the principle of assigning mass recursively to subsets of increasing size. In this case each BF b is represented by the vector:

$$b = [b(x), b(y), b(z), b(\{x,y\}), b(\{x,z\}), b(\{y,z\})]' \in \mathbb{R}^6.$$

If the mass not assigned to singletons $1 - m_b(x) - m_b(y) - m_b(z)$ is attributed to $A = \Theta$, we have that $b(\{x,y\}) = m_b(\{x,y\}) + m_b(x) + m_b(y) = m_b(x) + m_b(y)$ and so on for all size-2 events, so that b belongs to the three-dimensional region:

$$\mathcal{D} = \Big\{ b \;:\; 0 \leq b(x) + b(y) + b(z) \leq 1, b(\{x,y\}) = b(x) + b(y), \\ b(\{x,z\}) = b(x) + b(z), b(\{y,z\}) = b(y) + b(z) \Big\}. \tag{6}$$

It is easy to realize that any arbitrary belief function $b \in \mathcal{B}_3$ on Θ_3 can be mapped onto a point $\pi[b]$ of $\mathcal{D}$

$$\pi[b] = \Big[b(x), b(y), b(z), b(x) + b(y), b(x) + b(z), b(y) + b(z) \Big]' \in \mathcal{D} \tag{7}$$

through a projection map $\pi : \mathcal{B} \to \mathcal{D}$. Let us call $\mathcal{D}$ the "base" of $\mathcal{B}_3$. Such base admits as coordinate chart the basic probabilities of the singletons, as each point $d \in \mathcal{D}$ can be written as: $d = [m_b(x), m_b(y), m_b(z)]'$.

Given a point $d \in \mathcal{D}$ on the base, we might want to understand what belief functions $b \in \mathcal{B}_3$ have d as projection: $\mathcal{F}(d) \doteq \{b : \pi[b] = d\}$. In virtue of the constraint acting on b.p.a.s, such belief functions have to meet the following constraints:

$$\begin{cases} m_b(\{x,y\}) \geq 0 \equiv b(\{x,y\}) \geq b(x) + b(y) \;\equiv\; b(\{x,y\}) \geq m_b(x) + m_b(y) \\ m_b(\{x,z\}) \geq 0 \equiv b(\{x,z\}) \geq b(x) + b(z) \;\equiv\; b(\{x,z\}) \geq m_b(x) + m_b(z) \\ m_b(\{y,z\}) \geq 0 \equiv b(\{y,z\}) \geq b(y) + b(z) \;\equiv\; b(\{y,z\}) \geq m_b(y) + m_b(z) \\ m_b(\Theta) \geq 0 \equiv \\ \qquad b(\Theta) + b(x) + b(y) + b(z) \geq b(\{x,y\}) + b(\{x,z\}) + b(\{y,z\}) \\ \qquad \equiv b(\{x,y\}) + b(\{x,z\}) + b(\{y,z\}) \leq 1 + b(x) + b(y) + b(z) \\ \qquad \equiv b(\{x,y\}) + b(\{x,z\}) + b(\{y,z\}) \leq 1 + m_b(x) + m_b(y) + m_b(z), \end{cases}$$

where $\equiv$ denotes equivalence.

Each BF $d \in \mathcal{D}$ on the base is associated with a whole "fiber" $\mathcal{F}(d)$ of belief functions projecting onto d (as they have the same b.p.a. on sigletons):

$$\mathcal{F}(d) = \Big\{ b \in \mathcal{B}_3 : \; b(x) = m_b(x), b(y) = m_b(y), b(z) = m_b(z), \\ b(\{x,y\}) \geq m_b(x) + m_b(y), \\ b(\{x,z\}) \geq m_b(x) + m_b(z), b(\{y,z\}) \geq m_b(y) + m_b(z) \Big\}. \tag{8}$$

Belief functions on $\mathcal{F}(d)$ can be parameterized by the three coordinates $m_b(\{x, y\})$, $m_b(\{x, z\})$ and $m_b(\{y, z\})$ corresponding to the basic probabilities of events of size greater than 1 (see Figure 2-right).

Given the nature of b.p.a.s as simplicial coordinates (4) we can infer that the base (6) of the belief space $\mathcal{B}_3$ is in fact the three dimensional simplex $\mathcal{D} = Cl(b_x, b_y, b_z, b_\Theta)$, as all the mass is distributed among $\{x\}, \{y\}, \{z\}$ and Θ in all possible ways (see Figure 2-left). The base $\mathcal{D}$ is the simplex of all *quasi Bayesian* belief functions on Θ_3, i.e., the belief functions for which $m_b(A) \neq 0$ iff $|A| = 1, n$ [15]. For each point d of the base the corresponding fiber (8) is also a simplex of dimension 3 (Figure 2-right).

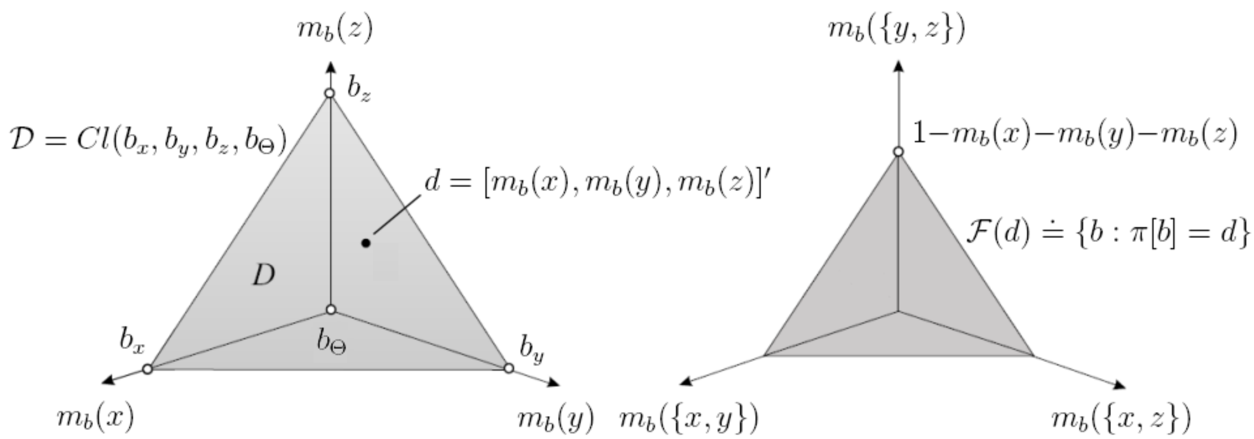

Fig. 2. Bundle structure of the belief space in the case of ternary frames $\Theta_3 = \{x, y, z\}$.

Summarizing, we have learned that (at least in the ternary case):

- the belief space can be decomposed into a base, i.e., the set of BFs assigning mass zero to events A of size $1 < |A| < n$,

$$\mathcal{D} \doteq \{b \in \mathcal{B} : m_b(A) = 0, \forall\ A : 1 < |A| < n\}, \tag{9}$$

  and a number of fibers $\mathcal{F}(d)$ passing each through a point d of the base;
- points on the base are parameterized by the masses assigned to singletons $d = [m_b(A), |A| = 1]'$, while points on the fibers have as coordinates the massed assigned to higher size events, $[m_b(A), 1 < |A| < n]'$;
- both base and fiber are simplices.

As we will see in the following, the same sort of decomposition applies to general belief spaces recursively for increasing size of the events we assign mass to. We first need to introduce the necessary mathematical tool.

B. Definition of smooth fiber bundles

Fiber bundles [11] generalize of the notion of Cartesian product.

Definition 3. *A smooth fiber bundle ξ is a composed object $\{E, B, \pi, F, G, \mathcal{U}\}$, where*

1) E *is an* $s+r$*-dimensional differentiable manifold called* total space*;*
2) B *is an* r*-dimensional differentiable manifold called* base space*;*
3) F *is an* s*-dimensional differentiable manifold called* fiber*;*
4) $\pi : E \rightarrow B$ *is a smooth application of full rank* r *in each point of* B*, called* projection*;*
5) G *is the* structure group*;*
6) *the atlas* $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ *defines a* bundle structure*; namely*
   - *the base* B *admits a covering with open sets* U_α *such that*
   - $E_\alpha \doteq \pi^{-1}(U_\alpha)$ *is equipped with* smooth direct product coordinates

$$\begin{array}{rccc} \phi_\alpha : & \pi^{-1}(U_\alpha) & \rightarrow & U_\alpha \times F \\ & e & \mapsto & (\phi'_\alpha(e), \phi''_\alpha(e)) \end{array} \tag{10}$$

   satisfying two conditions:

   – *the coordinate component with values in the base space is* compatible with the projection map*:*

$$\pi \circ \phi_\alpha^{-1}(x, f) = x \tag{11}$$

   or equivalently $\phi'_\alpha(e) = \pi(e)$*;*

   – *the coordinate component with values on the fiber can be transformed, jumping from a coordinate chart into another, by means of elements of the structure group. Formally the applications*

$$\begin{array}{rccc} \lambda_{\alpha\beta} \doteq \phi_\beta \phi_\alpha^{-1} : & U_{\alpha\beta} \times F & \rightarrow & U_{\alpha\beta} \times F \\ & (x, f) & \mapsto & (x, T^{\alpha\beta}(x) f) \end{array}$$

   called gluing functions *are implemented by means of transformations* $T^{\alpha\beta}(x) : F \rightarrow F$ *defined by applications from a domain* $U_{\alpha\beta}$ *to the structure group*

$$T^{\alpha\beta} : U_{\alpha\beta} \rightarrow G$$

satisfying the following conditions

$$T^{\alpha\beta} = (T^{\beta\alpha})^{-1}, \qquad T^{\alpha\beta}T^{\beta\gamma}T^{\gamma\alpha} = 1. \tag{12}$$

Intuitively, the base space is covered by a number of open neighborhoods $\{U_\alpha\}$, which induce a similar covering $\{E_\alpha = \pi^{-1}(U_\alpha)\}$ on the total space E. Points e of each neighborhood E_α of the total space admit coordinates separable into two parts: the first one $\phi'(e) = \pi(e)$ is the projection of e onto the base B, while the second part is its coordinate on the fiber F. Fiber coordinates are such that in the intersection of two different charts $E_\alpha \cap E_\beta$ they can be transformed into each other by means of the action of a group G.

In the following, however, all the involved manifolds will be linear spaces, so that each of them can be covered by a single chart. This makes the bundle structure trivial, i.e., the identity transformation. The reader can then safely ignore the above conditions on ϕ''_α.

C. Points of the Cartesian space as sum functions

As the belief space does not exhaust the whole $\mathbb{R}^{N-2}$ it is natural to wonder whether arbitrary points of $\mathbb{R}^{N-2}$, possibly "outside" $\mathcal{B}$, have any meaningful interpretation in this framework [8]. In fact, each vector $v = [v_A, \emptyset \subsetneq A \subseteq \Theta]' \in \mathbb{R}^{N-1}$ can be thought of as a set function $\varsigma : 2^\Theta \setminus \emptyset \to \mathbb{R}$ s.t. $\varsigma(A) = v_A$. By applying the Möbius transformation (1) to such functions ς we obtain another set function $m_\varsigma : 2^\Theta \setminus \emptyset \to \mathbb{R}$ such that $\varsigma(A) = \sum_{B \subseteq A} m_\varsigma(B)$. In other words each vector ς of $\mathbb{R}^{N-1}$ can be thought of as a sum function. However, contrarily to basic probability assignments, the Möbius inverses m_ς of generic sum functions $\varsigma \in \mathbb{R}^{N-1}$ are not guaranteed to meet the non-negativity constraint: $m_\varsigma(A) \ngeq 0 \; \forall A \subseteq \Theta$.

Now, the section $\{v \in \mathbb{R}^{N-1} : v_\Theta = 1\}$ of $\mathbb{R}^{N-1}$ corresponds to the constraint $\varsigma(\Theta) = 1$. Therefore, all the points of this section are sum functions meeting the normalization axiom $\sum_{A \subset \Theta} m_\varsigma(A) = 1$ or *normalized sum functions* (NSFs). Normalized sum functions are the natural extensions of belief functions in the geometric framework.

D. Recursive bundle structure of the space of sum functions

We can now reinterpret our analysis of the ternary case by means of the formal definition 3 of smooth fiber bundle. The belief space $\mathcal{B}_3$ can be in fact equipped with a base (6), and a projection (7) from the total space $\mathcal{R}^6$ to the base, which generates fibers of the form (8).

However, the original definition of fiber bundle requires the involved spaces to be *manifolds*, while the ternary case suggests we have here to deal with *simplices*.
We can notice though how the idea of recursively assigning mass to subsets of increasing size does not necessarily require the mass itself to be positive. In other words, this procedure can be in fact applied to normalized sum functions, yielding a classical fiber bundle structure for the space $\mathcal{S} = \mathbb{R}^{N-2}$ of all NSFs on Θ, in which all the involved bases and fibers are *linear* spaces. We will see in the following how this has to be modified when considering proper belief functions.

Theorem 1. *The space $\mathcal{S} = \mathbb{R}^{N-2}$ of all the sum functions ς with domain on a finite frame Θ of cardinality $|\Theta| = n$ has a recursive fiber bundle structure, i.e., there exists a sequence of smooth fiber bundles*

$$\xi_i = \left\{\mathcal{F}_{\mathcal{S}}^{(i-1)}, \mathcal{D}_{\mathcal{S}}^{(i)}, \mathcal{F}_{\mathcal{S}}^{(i)}, \pi_i\right\}, \qquad i = 1, ..., n-1$$

where $\mathcal{F}_{\mathcal{S}}^{(0)} = \mathcal{S} = \mathbb{R}^{N-2}$, the total space *$\mathcal{F}_{\mathcal{S}}^{(i-1)}$, the* base space *$\mathcal{D}_{\mathcal{S}}^{(i)}$ and the* fiber *$\mathcal{F}_{\mathcal{S}}^{(i)}$ of the i-th bundle level are linear subspaces of $\mathbb{R}^{N-2}$ of dimensions $\sum_{k=i}^{n-1}\binom{n}{k}$, $\binom{n}{i}$, $\sum_{k=i+1}^{n-1}\binom{n}{k}$ respectively. Both $\mathcal{F}_{\mathcal{S}}^{(i-1)}$ and $\mathcal{D}_{\mathcal{S}}^{(i)}$ admit a* global *coordinate chart. As*

$$\dim \mathcal{F}_{\mathcal{S}}^{(i-1)} = \sum_{k=i,\dots,n-1} \binom{n}{k} = \left|\left\{A \subset \Theta : i \leq |A| < n\right\}\right|,$$

each point ς^{i-1} of $\mathcal{F}_{\mathcal{S}}^{(i-1)}$ can be written as

$$\varsigma^{i-1} = \left[\varsigma^{i-1}(A), A \subset \Theta, i \leq |A| < n\right]'$$

and the smooth direct product coordinates *(10) at the i-th bundle level are*

$$\phi'(\varsigma^{i-1}) = \left\{\varsigma^{i-1}(A),\ |A| = i\right\}, \quad \phi''(\varsigma^{i-1}) = \left\{\varsigma^{i-1}(A),\ i < |A| < n\right\}.$$

The projection map π_i of the i-th bundle level is a full-rank differentiable application

$$\begin{array}{rccc} \pi_i : & \mathcal{F}_{\mathcal{S}}^{(i-1)} & \rightarrow & \mathcal{D}_{\mathcal{S}}^{(i)} \\ & \varsigma^{i-1} & \mapsto & \pi_i[\varsigma^{i-1}] \end{array}$$

whose expression in this coordinate chart is

$$\pi_i[\varsigma^{i-1}] = [\varsigma^{i-1}(A), |A| = i]'. \tag{13}$$

Proof:

The bottom line of the proof is that *the mass associated with a sum function can be recursively assigned to subsets of increasing size*. We prove Theorem 1 by induction.

First level of the bundle structure. As we mentioned above, each normalized sum function $\varsigma \in \mathbb{R}^{N-2}$ is uniquely associated to a mass function m_ς through the inversion lemma. To define a base space of the first level, we set to zero the mass of all events of size $1 < |A| < n$. This determines a linear space $\mathcal{D}_\mathcal{S} \subset \mathcal{S} = \mathbb{R}^{N-2}$ defined by the system of linear equations

$$\mathcal{D}_\mathcal{S} \doteq \Big\{ \varsigma \in \mathbb{R}^{N-2} : m_\varsigma(A) = \sum_{B \subset A} (-1)^{|A-B|} \varsigma(B) = 0,\ 1 < |A| < n \Big\}$$

of dimension $\dim \mathcal{D}_\mathcal{S} = n = |\Theta|$ (as there are n unconstrained variables corresponding to the singletons). As $\mathcal{D}_\mathcal{S}$ is linear it admits a global coordinate chart. Each point $d \in \mathcal{D}$ is parameterized by the masses the corresponding sum function ς assigns to singletons

$$d = [m_\varsigma(A) = \varsigma(A), |A| = 1]'.$$

The second step is to precise a projection map between the total space $\mathcal{S} = \mathbb{R}^{N-2}$ and the base $\mathcal{D}_\mathcal{S}$. The Moebius equation induces indeed a projection map from $\mathcal{S}$ to $\mathcal{D}_\mathcal{S}$

$$\begin{array}{rccc} \pi : & \mathcal{S} = \mathbb{R}^{N-2} & \rightarrow & \mathcal{D}_\mathcal{S} \subset \mathbb{R}^{N-2} \\ & \varsigma & \mapsto & \pi[\varsigma] \end{array}$$

mapping each NSF $\varsigma \in \mathbb{R}^{N-2}$ to a point $p[\varsigma]$ of the base space $\mathcal{D}$:

$$\pi[\varsigma](A) = [\varsigma(A), |A| = 1]'. \tag{14}$$

Finally, to define a bundle structure we need to describe the fibers of the total space $\mathcal{S} = \mathbb{R}^{N-2}$, i.e., the vector subspaces of $\mathbb{R}^{N-2}$ which project on a given point $d \in \mathcal{D}$ of the base.
Each point $d \in \mathcal{D}$ is indeed associated with the linear space of all the NSFs $\varsigma \in \mathbb{R}^{N-2}$ whose projection $p[\varsigma]$ on $\mathcal{D}$ is d:

$$\mathcal{F}_\mathcal{S}(d) \doteq \Big\{ \varsigma \in \mathcal{S} : \pi[\varsigma] = d \in \mathcal{D} \Big\}.$$

It is easy to see that as d varies on the base space $\mathcal{D}$, the linear spaces we obtain are all diffeomorphic to $\mathcal{F} \doteq \mathbb{R}^{N-2-n}$.

According Definition 3 this defines a bundle structure, since:

- $E \doteq \mathcal{S} = \mathbb{R}^{N-2}$ is a smooth manifold, in particular a linear space;
- $B \doteq \mathcal{D}_\mathcal{S}$, the base space, is a smooth (linear) manifold;
- $F = \mathcal{F}_\mathcal{S}$, the fiber, is a smooth manifold, again a linear space.

Finally, the projection $\pi : \mathcal{S} = \mathbb{R}^{N-2} \rightarrow \mathcal{D}_\mathcal{S}$ is differentiable (as it is a linear function of the coordinates $\varsigma(A)$ of ς) and has full rank n in every point $\varsigma \in \mathbb{R}^{N-2}$. This is easy to see when representing π as a matrix (ς is a vector, hence a linear function of ς can always thought of as a matrix)

$$\pi[\varsigma] = \Pi\varsigma$$

where

$$\Pi = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ & & \cdots & & & \cdots & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} = [I_n | 0_{n\times(N-2-n)}]$$

according to Equation (14), and the rows of Π are obviously linearly independent.

As mentioned above the bundle structure (Definition 3, (6)) is trivial, since $\mathcal{D}_\mathcal{S}$ is linear and can be covered by a single coordinate system (10). The direct product coordinates are

$$\begin{array}{rccc} \phi : & \mathcal{S} = \mathbb{R}^{N-2} & \rightarrow & \mathcal{D}_\mathcal{S} \times \mathcal{F}_\mathcal{S} \\ & \varsigma & \mapsto & (\pi[\varsigma], f[\varsigma]) \end{array}$$

where the coordinates of ς on the fiber $\mathcal{F}_\mathcal{S}$ are the masses it assigns to higher size events

$$f[\varsigma] = [m_\varsigma(A), 1 < |A| < n]'.$$

Bundle structure of level i.

By induction, let us suppose that $\mathcal{S}$ admits a recursive bundle structure for all sizes from 1 to $i-1$ characterized according to the hypotheses, and prove that $\mathcal{F}_\mathcal{S}^{(i-1)}$ can in turn be decomposed in the same way into a linear base space and a collection of diffeomorphic fibers. By inductive hypothesis $\mathcal{F}_\mathcal{S}^{(i-1)}$ has dimension $N-2-\sum_{k=1}^{i-1}\binom{n}{k}$ and each point $\varsigma^{i-1} \in \mathcal{F}_\mathcal{S}^{(i-1)}$ has coordinates[3]

$$\varsigma^{i-1} = [\varsigma^{i-1}(A), i \leq |A| < n]'.$$

We can therefore apply the constraint $\varsigma^{i-1}(A) = 0$, $i < |A| < n$ which identifies the linear variety

$$\mathcal{D}_\mathcal{S}^{(i)} \doteq \left\{ \varsigma^{i-1} \in \mathcal{F}_\mathcal{S}^{(i-1)} : \varsigma^{i-1}(A) = 0,\ i < |A| < n \right\} \quad (15)$$

[3]The quantity $\varsigma^{i-1}(A)$ is in fact the mass $m_\varsigma(A)$ the original NSF ς attaches to A, but this is irrelevant for the purpose of the decomposition.

embedded in $\mathcal{F}_{\mathcal{S}}^{(i-1)}$, of dimension $\binom{n}{i}$ (the number of size-i subsets of Θ).

The projection map (13) induces in $\mathcal{F}_{\mathcal{S}}^{(i-1)}$ fibers of the form

$$\mathcal{F}_{\mathcal{S}}^{(i)} \doteq \{\varsigma^{i-1} \in \mathcal{F}_{\mathcal{S}}^{(i-1)} : \pi_i[\varsigma^{i-1}] = const\}$$

which are also linear manifolds, and induce a trivial bundle structure in $\mathcal{F}_{\mathcal{S}}^{(i-1)}$

$$\begin{array}{rccc} \phi : & \mathcal{F}_{\mathcal{S}}^{(i-1)} & \rightarrow & \mathcal{D}_{\mathcal{S}}^{(i)} \times \mathcal{F}_{\mathcal{S}}^{(i)} \\ & \varsigma^{i-1} & \mapsto & (\phi'(\varsigma^{i-1}), \phi''(\varsigma^{i-1})) \end{array}$$

with $\phi'(\varsigma^{i-1}) = \pi_i[\varsigma^{i-1}] = [\varsigma^{i-1}(A), |A| = i]'$. Again, the map (13) is differentiable and has full rank for its $\binom{n}{i}$ rows are independent.

The decomposition ends when $\dim \mathcal{F}_{\mathcal{S}}^{(n)} = 0$, and all fibers reduce to points of $\mathcal{S}$. ■

Bases and fibers are simply geometric counterparts of the mass assignment mechanism. Having assigned a certain amount of mass to subsets of size smaller than i, the fraction of mass attributed to size-i subsets determines a point on a linear space: $\mathcal{D}_{\mathcal{S}}^{(i)}$. For each point of $\mathcal{D}_{\mathcal{S}}^{(i)}$ the remaining mass can "float" among the higher size subsets, describing again a vector space $\mathcal{F}_{\mathcal{S}}^{(i)}$.

E. Recursive bundle structure of the belief space

As we have seen in the ternary example of Section III-A, as the belief space is a simplex immersed in $\mathcal{S} = \mathbb{R}^{N-2}$, the fibers of $\mathbb{R}^{N-2}$ do intersect the space of belief functions too. $\mathcal{B}$ then inherits some sort of bundle structure from the Cartesian space in which it is immersed. The belief space can also be recursively decomposed into fibers associated with events A of the same size. As one can easily conjecture, the intersections fibers of $\mathbb{R}^{N-2}$ with the simplex $\mathcal{B}$ are themselves simplices: bases and fibers in the case of the belief space are therefore polytopes instead of linear spaces. Due to the decomposition of $\mathbb{R}^{N-2}$ into basis and fibers, we can apply the positivity and normalization constraints which distinguish belief functions from NSFs *separately at each level*, eliminating at each step the fibers passing through points of the base that do not meet these conditions.

We first need a simple combinatorial result.

Lemma 1. $\displaystyle\sum_{|A|=i} b(A) \leq 1 + \sum_{m=1}^{i-1} (-1)^{i-(m+1)} \binom{n-(m+1)}{i-m} \cdot \sum_{|B|=m} b(B)$*, and the upper bound is reached when* $\sum_{|A|=i} m_b(A) = 1 - \sum_{|A|<i} m_b(A)$.

Proof: Since $\binom{n-m}{i-m}$ is the number of subsets of size i containing a fixed set B, $|B| = m$ in a frame with n elements, we can write:

$$\begin{aligned}\sum_{|A|=i} b(A) &= \sum_{|A|=i}\sum_{B\subseteq A} m_b(B) = \sum_{m=1}^{i}\sum_{|B|=m}\binom{n-m}{i-m} m_b(B) \\ &= \sum_{|B|=i} m_b(B) + \sum_{m=1}^{i-1}\sum_{|B|=m}\binom{n-m}{i-m} m_b(B) \\ &\leq 1 - \sum_{|B|<i} m_b(B) + \sum_{m=1}^{i-1}\sum_{|B|=m}\binom{n-m}{i-m} m_b(B),\end{aligned} \tag{16}$$

as $\sum_{|B|=i} m_b(B) = 1 - \sum_{|B|<i} m_b(B)$ by normalization. By Möbius inversion (1):

$$\begin{aligned}\sum_{|A|<i} m_b(A) &= \sum_{|A|<i}\sum_{B\subseteq A}(-1)^{|A-B|} b(B) \\ &= \sum_{|A|=m=1}^{i-1}\sum_{|B|=l=1}^{m}(-1)^{m-l}\binom{n-l}{m-l}\sum_{|B|=l} b(B)\end{aligned} \tag{17}$$

for, again, $\binom{n-l}{m-l}$ is the number of subsets of size m containing a fixed set B, $|B| = l$ in a frame with n elements. The role of the indexes m and l can be exchanged, obtaining:

$$\sum_{|B|=l=1}^{i-1} m_b(B) = \sum_{|B|=l=1}^{i-1}\left[\sum_{|B|=l} b(B)\cdot\sum_{m=l}^{i-1}(-1)^{m-l}\binom{n-l}{m-l}\right]. \tag{18}$$

Now, a well known combinatorial identity ([**?**], volume 3, Equation (1.9)) states that, for $i - (l+1) \geq 1$:

$$\sum_{m=l}^{i-1}(-1)^{m-l}\binom{n-l}{m-l} = (-1)^{i-(l+1)}\binom{n-(l+1)}{i-(l+1)}. \tag{19}$$

By applying (19) to the last equality, (17) becomes:

$$\sum_{|B|=l=1}^{i-1}\left[\sum_{|B|=l} b(B)\cdot(-1)^{i-(l+1)}\binom{n-(l+1)}{i-(l+1)}\right]. \tag{20}$$

Similarly, by (18) we have:

$$\begin{aligned}\sum_{m=1}^{i-1}\sum_{|B|=m}\binom{n-m}{i-m} m_b(B) &= \sum_{l=1}^{i-1}\sum_{|B|=l} b(B)\cdot\sum_{m=l}^{i-1}(-1)^{m-l}\binom{n-l}{m-l}\binom{n-m}{i-m} \\ &= \sum_{l=1}^{i-1}\sum_{|B|=l} b(B)\cdot\sum_{m=l}^{i-1}(-1)^{m-l}\binom{i-l}{m-l}\binom{n-l}{i-l},\end{aligned}$$

as it is easy to verify that $\binom{n-l}{m-l}\binom{n-m}{i-m} = \binom{i-l}{m-l}\binom{n-l}{i-l}$.
By applying (19) again to the last equality we get:

$$\sum_{m=1}^{i-1}\sum_{|B|=m}\binom{n-m}{i-m}m_b(B) = \sum_{l=1}^{i-1}\sum_{|B|=l}(-1)^{i-(l+1)}\binom{n-l}{i-l}. \tag{21}$$

By replacing (18) and (21) in (16) we get the thesis. ∎

The bottom line of Lemma 1 is that, given a mass assignment for events of size $1, ..., i-1$ the upper bound for $\sum_{|A|=i} b(A)$ is obtained by assigning all the remaining mass to the collection of size i subsets.

Theorem 2. *The belief space $\mathcal{B} \subset \mathcal{S} = \mathbb{R}^{N-2}$ inherits by intersection with the recursive bundle structure of $\mathcal{S}$ a "convex"-bundle decomposition. Each i-th level "fiber" can be expressed as*

$$\mathcal{F}_{\mathcal{B}}^{(i-1)}(d^1, ..., d^{i-1}) = \Big\{ b \in \mathcal{B} : V_i \wedge \cdots \wedge V_{n-1}(d^1, ..., d^{i-1}) \Big\}, \tag{22}$$

where $V_i(d^1, ..., d^{i-1})$ denotes the system of constraints

$$V_i(d^1, ..., d^{i-1}) : \begin{cases} m_b(A) \geq 0 & \forall A \subseteq \Theta : |A| = i, \\ \displaystyle\sum_{|A|=i} m_b(A) \leq 1 - \sum_{|A|<i} m_b(A) \end{cases} \tag{23}$$

and depends on the mass assigned to lower size subsets $d^m = [m_b(A), |A| = m]'$, $m = 1, ..., i-1$. The corresponding "base" $\mathcal{D}_{\mathcal{B}}^{(i)}(d^1, ..., d^{i-1})$ is expressed in terms of basic probability assignments as the collection of BFs $b \in \mathcal{F}^{(i-1)}(d^1, ..., d^{i-1})$ such that

$$\begin{cases} m_b(A) = 0, & \forall A : i < |A| < n \\ m_b(A) \geq 0, & \forall A : |A| = i \\ \displaystyle\sum_{|A|=i} m_b(A) \leq 1 - \sum_{|A|<i} m_b(A). \end{cases} \tag{24}$$

Proof: To understand the effect on $\mathcal{B} \subset \mathcal{S}$ of the bundle decomposition of the space of normalized sum functions $\mathcal{S} = \mathbb{R}^{N-2}$ in which it is immersed we need to consider the effect of the positivity $m_\varsigma \geq 0$ and normalization $\sum_A m_\varsigma(A) = 1$ conditions. They constrain the admissible values of the coordinates of points of $\mathcal{S}$.
We can appreciate how these constraints are separable into groups that apply to subsets of the same size. The set of conditions

$$\begin{cases} m_b(A) \geq 0, & \forall A \subseteq \Theta \\ \sum_{A\subseteq\Theta} m_b(A) = 1 \end{cases}$$

can in fact be decomposed as $V_1 \wedge \cdots \wedge V_{n-1}$, where the system of constraints V_i is given by Equation (23). The bottom inequality in (23) implies that, given a mass assignment for events of size $1, ..., k-1$ ($\sum_{|A|<k} m_b(A)$) the upper bound for $\sum_{|A|=k} b(A)$ is obtained by assigning all the remaining mass to the collection of size k subsets: $\sum_{|A|=k} m_b(A) = 1 - \sum_{|A|<k} m_b(A)$ (Lemma 1). Let us see their effect on the bundle structure of $\mathcal{S}$.

Level 1. By definition $\mathcal{B} = \{\varsigma \in \mathcal{S} : V_1 \wedge \cdots \wedge V_{n-1}\}$. As the coordinates of the points of $\mathcal{S}$ are decomposed into coordinates on the base $[m_\varsigma(A), |A| = 1]'$ and coordinates on the fiber $[m_\varsigma(A), 1 < |A| < n]'$ it is easy to see that V_1

$$\begin{cases} m_b(A) \geq 0, & |A| = 1 \\ \sum_{|A|=1} m_b(A) \leq 1 \end{cases} \tag{25}$$

acts only on the base $\mathcal{D}^{(1)}_{\mathcal{S}}$, yielding a new set

$$\mathcal{D}^{(1)}_{\mathcal{B}} = \Big\{ \quad b \in \mathcal{B} : m_b(A) = 0 \;\; 1 < |A| < n, m_b(A) \geq 0 \;\; |A| = 1, \sum_{|A|=1} m_b(A) \leq 1 \Big\}$$

of the form of Equation (24) for $i = 1$.

This in turn selects the fibers of $\mathcal{S}$ passing through $\mathcal{D}^{(1)}_{\mathcal{B}}$, and discards the others. As a matter of fact there cannot be admissible belief functions within fibers passing through points outside this region, since all points of a fiber $\mathcal{F}^{(1)}_{\mathcal{S}}$ share the same level 1 coordinates d^1 : when the basis point d^1 does not meet the inequalities (25), none of them can.

Therefore the remaining constraints $V_2 \wedge \cdots \wedge V_{n-1}$ act on the fibers $\mathcal{F}^{(1)}_{\mathcal{S}}$ of $\mathcal{S}$ passing through $\mathcal{D}^{(1)}_{\mathcal{B}}$. However, Equation (23) shows that those higher size constraints $V_2, ..., V_{n-1}$ in fact depend on the point $d^1 = [m_b(A), |A| = 1]' \in \mathcal{D}^{(1)}_{\mathcal{B}}$ on the base space. Each admissible fiber $\mathcal{F}^{(1)}_{\mathcal{S}} \sim \mathbb{R}^{N-2-n}$ of $\mathcal{S}$ is then subject to a different system of constraints $V_2 \wedge \cdots \wedge V_{n-1}(d^1)$, yielding the corresponding first level fiber of $\mathcal{B}$ (see Equation (22))

$$\mathcal{F}^{(1)}_{\mathcal{B}}(d^1) = \Big\{ b \in \mathcal{F}^{(1)}_{\mathcal{S}}(d^1) : V_2 \wedge \cdots \wedge V_{n-1}(d^1) \Big\}.$$

Level i. Let us now suppose that, by induction, we have a family of constraints $V_i \wedge \cdots \wedge V_{n-1}(d^1, ..., d^{i-1})$ of the form of Equation (23), acting on $\mathcal{F}^{(i-1)}_{\mathcal{S}}(d^1, ..., d^{i-1})$.
The points of $\mathcal{F}^{(i-1)}_{\mathcal{S}}(d^1, ..., d^{i-1})$ have coordinates which can be decomposed into coordinates $d^i = [m_\varsigma(A), |A| = i]'$ on the base $\mathcal{D}^{(i)}_{\mathcal{S}}$ and coordinates on the fiber $\mathcal{F}^{(i)}_{\mathcal{S}}$. Again, the set of

constraints V_i acts on coordinates associated with size-i events only, i.e., it acts on $\mathcal{D}_{\mathcal{S}}^{(i)}$ and not on $\mathcal{F}_{\mathcal{S}}^{(i)}$.

Furthermore, constraints of type (23) for $k > i$ become trivial when acting on $\mathcal{D}_{\mathcal{S}}^{(i)}$. In fact, inequalities of the form $m_b(A) \geq 0$, $|A| > i$ are satisfied by $\mathcal{D}_{\mathcal{S}}^{(i)}$ by definition, since it imposes $m_\varsigma(A) = 0$, $|A| > i$. On the other side, all inequalities corresponding to the second row of Equation (23) for $k > i$ reduce to the corresponding inequality for size-i subsets. Instead of displaying a boring combinatorial proof, we can just recall the meaning of Lemma 1: the upper bound for $\sum_{|A|=i} b(A)$ is obtained by assigning maximal mass to the collection of size i subsets. But then, points in $\mathcal{D}_{\mathcal{S}}^{(i)}$ correspond to a zero-assignment for higher size events $m_\varsigma(A) = 0, i < |A| < n$ and all those upper bounds are automatically satisfied.

We then get the i-th level base for $\mathcal{B}$: $\mathcal{D}_{\mathcal{B}}^{(i)}(d^1, ..., d^{i-1})$ is the set of BFs $b \in \mathcal{F}_{\mathcal{S}}^{(i-1)}(d^1, ..., d^{i-1})$ such that conditions (24) are satisfied. The remaining constraints $V_{i+1} \wedge ... \wedge V_{n-1}(d^1, ..., d^i)$ act on the fibers $\mathcal{F}_{\mathcal{S}}^{(i)}$ of $\mathcal{S}$ passing through points d^i of $\mathcal{D}_{\mathcal{B}}^{(i)}(d^1, ..., d^{i-1})$, yielding a collection of level-i fibers for $\mathcal{B}$: $\mathcal{F}_{\mathcal{B}}^{(i)}(d^1, ..., d^i) = \left\{ b \in \mathcal{F}_{\mathcal{S}}^{(i)} : V_{i+1} \wedge ... \wedge V_{n-1}(d^1, ..., d^i) \right\}$. ■

F. *Bases and fibers as simplices*

Simplicial and bundle structure coexist in the space of belief functions, both of them consequences of the interpretation of belief functions as sum functions, and of the basic probability assignment machinery. It is then natural to conjecture that bases and fibers of the recursive bundle decomposition of $\mathcal{B}$ must also be simplices of some sort, as suggested by the ternary example (Section III-A).

Let us work recursively, and suppose we have already assigned a mass $k < 1$ to the subsets of size smaller than i:

$$m_b(A) = const = m_A, \quad |A| < i. \tag{26}$$

All the admissible BFs constrained by this mass assignment are then forced to live in the following $(i-1)$-th level fiber of $\mathcal{B}$:

$$\mathcal{F}_{\mathcal{B}}^{(i-1)}(d^1, ..., d^{i-1}), \qquad d^j = [m_A, |A| = j]'$$

which, as we have seen in the proof of Theorem 2, is a function of the mass assigned to lower-size events.

We have seen that such fiber $\mathcal{F}_{\mathcal{B}}^{(i-1)}(d^1,...,d^{i-1})$ admits a pseudo-bundle structure whose pseudo-base space is $\mathcal{D}_{\mathcal{B}}^{(i)}(d^1,\cdots,d^{i-1})$ given by Equation (24). Let us denote by $k = \sum_{|A|<i} m_A$ the total mass already assigned to lower size events, and call

$$\mathcal{P}^{(i)}(d^1,...,d^{i-1}) \doteq \Big\{ b \in \mathcal{F}_{\mathcal{B}}^{(i-1)}(d^1,...,d^{i-1}) : \sum_{|A|=i} m_b(A) = 1-k \Big\}$$
$$\mathcal{O}^{(i)}(d^1,...,d^{i-1}) \doteq \Big\{ b \in \mathcal{F}_{\mathcal{B}}^{(i-1)}(d^1,...,d^{i-1}) : m_b(\Theta) = 1-k \Big\}$$

the collections of belief functions on the fiber $\mathcal{F}_{\mathcal{B}}^{(i-1)}(d^1,...,d^{i-1})$ assigning all the remaining basic probability $1-k$ to subsets of size i or to Θ, respectively.

As the simplicial coordinates of a BF in $\mathcal{B}$ are given by its basic probability assignment (4), each belief function $b \in \mathcal{F}_{\mathcal{B}}^{(i-1)}(d^1,...,d^{i-1})$ on such a fiber can be written as:

$$\begin{array}{rcl} b & = & \displaystyle\sum_{A\subseteq\Theta} m_b(A) b_A = \sum_{|A|<i} m_A b_A + \sum_{|A|\geq i} m_b(A) b_A \\ & = & \displaystyle\frac{k}{k}\sum_{|A|<i} m_A b_A + \frac{1-k}{1-k}\sum_{|A|\geq i} m_b(A) b_A \\ & = & \displaystyle\frac{k}{\sum_{|A|<i} m_A}\sum_{|A|<i} m_A b_A + \frac{1-k}{\sum_{|A|\geq i} m_b(A)}\sum_{|A|\geq i} m_b(A) b_A. \end{array}$$

We can therefore define two new belief functions b_0, b' associated with any $b \in \mathcal{F}_{\mathcal{B}}^{(i-1)}(d^1,...,d^{i-1})$, with basic probability assignments

$$\begin{array}{rcl} m_{b_0}(A) & \doteq & \dfrac{m_A}{\sum_{|B|<i} m_B} \;\; |A|<i, \qquad m_{b_0}(A) = 0 \;\; |A| \geq i; \\ m_{b'}(A) & \doteq & \dfrac{m_b(A)}{\sum_{|B|\geq i} m_b(B)} \;\; |A|\geq i, \qquad m_{b'}(A) = 0 \;\; |A| < i \end{array}$$

respectively, and decompose b as follows:

$$b = k\sum_{|A|<i} m_{b_0}(A) b_A + (1-k)\sum_{|A|\geq i} m_{b'}(A) b_A = kb_0 + (1-k)b'$$

where $b_0 \in Cl(b_A : |A|<i)$, $b' \in Cl(b_A : |A| \geq i)$. As

$$\sum_{A\subseteq\Theta} m_{b_0}(A) = \sum_{|A|<i} m_{b_0}(A) = \frac{\sum_{|A|<i} m_A}{\sum_{|A|<i} m_A} = 1, \qquad \sum_{A\subseteq\Theta} m_{b'}(A) = 1$$

both b_0 and b' are indeed admissible BFs, b_0 assigning non-zero mass to subsets of size smaller than i only, b' assigning mass to subsets of size i or higher.

However, b_0 is the same for all the BFs on the fiber $\mathcal{F}_{\mathcal{B}}^{(i-1)}(d^1,...,d^{i-1})$, as it is determined by the mass assignment (26). The other component b' is instead free to vary in $Cl(b_A : |A| \geq i)$.

Hence, we get the following convex expressions for $\mathcal{F}_{\mathcal{B}}^{(i-1)}, \mathcal{P}^{(i)}$ and $\mathcal{O}^{(i)}$ (neglecting for sake of simplicity the dependence on $d^1, ..., d^{i-1}$ or, equivalently, on b_0):

$$\begin{array}{rcl} \mathcal{F}_{\mathcal{B}}^{(i-1)} & = & \Big\{ b = kb_0 + (1-k)b', b' \in Cl(b_A, |A| \geq i) \Big\} \\ & = & kb_0 + (1-k)Cl(b_A, |A| \geq i), \\ \mathcal{P}^{(i)} & = & kb_0 + (1-k)Cl(b_A : |A| = i), \\ \mathcal{O}^{(i)} & = & kb_0 + (1-k)b_\Theta. \end{array} \quad (27)$$

By definition the i-th base $\mathcal{D}_{\mathcal{B}}^{(i)}$ is the collection of BFs such that

$$m_b(A) = 0 \quad i < |A| < n, \qquad m_b(A) = const = m_A \quad |A| < i$$

so that points on $\mathcal{D}_{\mathcal{B}}^{(i)}$ are free to distribute the remaining mass to Θ or size i events only. Therefore we obtain the following convex expression for the i-th level base space $\mathcal{D}_{\mathcal{B}}^{(i)}$ of $\mathcal{B}$:

$$\begin{array}{rcl} \mathcal{D}_{\mathcal{B}}^{(i)} & = & kb_0 + (1-k)Cl(b_A : |A| = i \; or \; A = \Theta) \\ & = & Cl(kb_0 + (1-k)b_A : |A| = i \; or \; A = \Theta) \\ & = & Cl(kb_0 + (1-k)b_\Theta, kb_0 + (1-k)b_A : |A| = i) = Cl(\mathcal{O}^{(i)}, \mathcal{P}^{(i)}). \end{array}$$

In the ternary case of Section III-A we get:

$$\mathcal{O}^{(1)} = b_\Theta, \quad \mathcal{P}^{(1)} = \mathcal{P} = Cl(b_x, b_y, b_z), \quad \mathcal{D}^{(1)} = Cl(\mathcal{O}^{(1)}, \mathcal{P}^{(1)}).$$

The elements of the bundle decomposition possess a natural meaning in terms of belief. In particular, $\mathcal{P}^{(1)} = \mathcal{P}$ is the set of all the Bayesian belief functions, while $\mathcal{D}^{(1)}$ is the collection of all the *discounted* probabilities [19], i.e., belief functions of the form $(1-\epsilon)p + \epsilon b_\Theta$, with $0 \leq \epsilon \leq 1$ and $p \in \mathcal{P}$.

On the other hand, BFs assigning mass to events of cardinality smaller than a certain size i are called in the literature *i-additive belief functions* ([16]). It is clear that the set $\mathcal{P}^{(i)}$ (27) is nothing but the collection of all i-additive BFs. The i-th level base of $\mathcal{B}$ can then be interpreted as the region of all "discounted" i-additive belief functions.

IV. Conclusions

In this paper we presented a novel description of the space of belief functions in terms of differential geometric notions. After giving an informal presentation of the way the b.p.a. mechanism induces a recursive decomposition of $\mathcal{B}$ we analyzed the simple case study of a ternary frame to get some intuition on the problem, and used it to prove the recursive bundle structure of the space of all sum functions, i.e., belief functions whose basic probability assignment does not meet the non-negativity constraint. As the belief space $\mathcal{B}$ is immersed in this Cartesian space it inherits a "pseudo" bundle structure in which bases and fibers are no more vector spaces but simplices in their own right, and possess interpretations in terms of i-additive belief functions.

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