(Submodular) Valued Constraints

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CSP

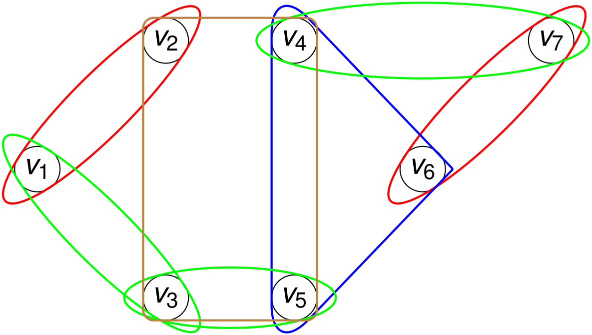
- CONSTRAINT SATISFACTION PROBLEM
- general framework from AI
- CSP $= \langle V, D, \mathcal{C} \rangle$
- (finite) **V**ariables, (finite) **D**omain, **C**onstraints
- constraint = relation on a subset of variables (scope)
- classical example: 3-colouring
  $G = \langle V, E \rangle \Rightarrow \langle V, \{R, G, B\}, \{\langle u \neq v \rangle \mid (u, v) \in E\} \rangle$

VCSP

- VALUED CONSTRAINT SATISFACTION PROBLEM
- not only **hard constraints** (relations)
- but also **soft constraints** ($\mathbb{Q}_+$)
- VCSP $= \langle V, D, \mathcal{C} \rangle$
- (finite) **V**ariables, (finite) **D**omain, **C**onstraints
- constraint = cost function: scope $\rightarrow \mathbb{Q}_+ \cup \{\infty\}$
- goal: minimise the total cost
  (=objective function=sum of all constraints)

“Hypergraph Min-Cost Colouring”

variables = vertices, constraints = coloured directed edges
goal: minimise the total cost



VCSP Examples

- example: 3-colouring with minimum discrepancy
- example: k-colouring as scheduling (weighted edges)
- VCSP is NP-hard
- structural restrictions (tree-like $\Rightarrow$ tractable)
- language restrictions (forms of constraints allowed)
- VCSP(L), k-colouring in VCSP($\{\neq\}$)
- VCSP($L_{\{0,\infty\}}$) $=$ CSP
- (DI)GRAPH (LIST/MIN-COST) HOMOMORPHISM
  ((sym) binary relation + unary relations/cost functions)

Modelling in VCSP

- properties & (dis)advantages of VCSP
- modelling problems in VCSP
- e-modelling for **polynomial-time** solvable problems (similar to the concept of P-completeness)

Theorem [Ž., Jeavons '07, in progress]

- (s, t)-MIN-CUT, SFM, and some other problems can be e-modelled efficiently by VCSP(L) for a tractable L
- MIN-CUT, SSFM can be e-modelled by VCSP(L) only for intractable Ls
- MIN-CUT can be modelled by VCSP with infinite domain

- hybrid reasons for tractability

Expressibility Example

values for variables listed in the scope of a constraint are constrained **explicitly**; due to the combined effect of constraints, **any** subset of variables is constrained **implicitly**

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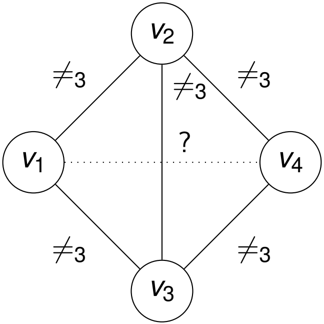

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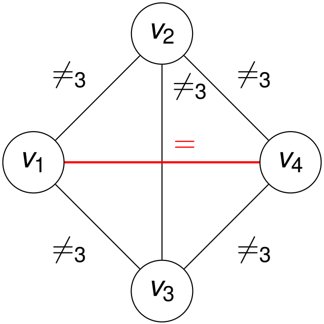

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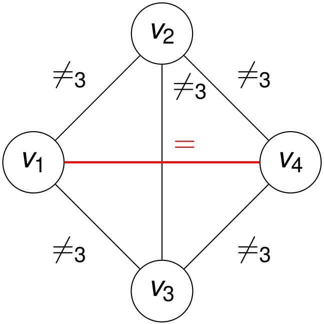
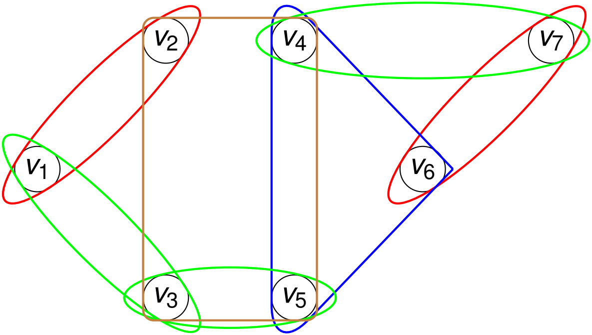


relation $=$ is **expressible** over $\{\neq_3\}$

Expressibility Example 2

instance $\mathcal{I} = \langle V, D, \mathcal{C} \rangle$ of VCSP(L)



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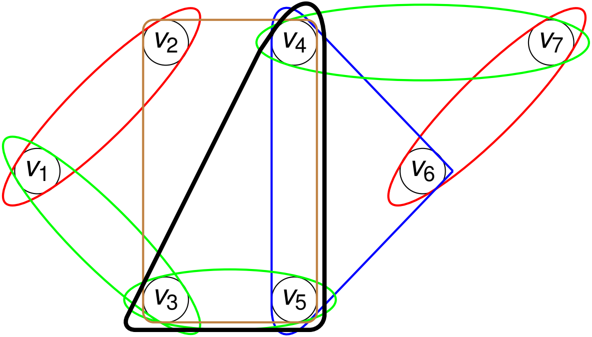

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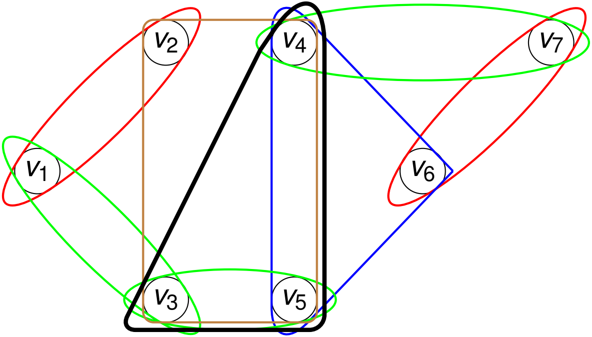

for any assignment of $l = \langle v_3, v_4, v_5 \rangle$ minimise over $V \setminus l$
gadget = instance + list of variables

Expressive Power

- given a VCSP(L) instance $\mathcal{I} = \langle\{x_1, \ldots, x_n\}, D, \mathcal{C}\rangle$ and a list $l = \langle v_1, \ldots, v_k\rangle$ of variables, the **projection** of $\mathcal{I}$ onto l is the minimum cost value c such that $\exists x_{k+1} \ldots x_n \mathcal{C}(x_1, \ldots, x_n)$ with total cost c
- ϕ is **expressible** over L if ϕ is equal to the projection of some instance of VCSP(L) onto some list of variables
- **expressive power** of L, $\langle L\rangle$
- known: VCSP(L) $\equiv_{\log}$ VCSP($\langle L\rangle$)

Fixed-Arity Languages

- **R**elations, **F**inite-valued and **G**eneral cost-functions
- of arity $\leq m$ over a domain of size d: $\mathbf{R}_{d,m}$, $\mathbf{F}_{d,m}$, $\mathbf{G}_{d,m}$

Theorem [Cohen, Jeavons, Ž. CP'07 & TCS'08]

- $\langle \mathbf{R}_{2,1} \rangle \subsetneq \langle \mathbf{R}_{2,2} \rangle \subsetneq \langle \mathbf{R}_{2,3} \rangle = \mathbf{R}_2$
- $\forall d \geq 3 : \langle \mathbf{R}_{d,1} \rangle \subsetneq \langle \mathbf{R}_{d,2} \rangle = \mathbf{R}_d$
- $\forall d \geq 2 : \langle \mathbf{F}_{f,1} \rangle \subsetneq \langle \mathbf{F}_{f,2} \rangle = \mathbf{F}_f$
  $\langle \mathbf{G}_{f,1} \rangle \subsetneq \langle \mathbf{G}_{f,2} \rangle = \mathbf{G}_f$

Fixed-Arity Max-Closed Languages

- ϕ of arity k is **max-closed** $\Leftrightarrow \forall u, v \in D^k$,

$$u \leq_{coordinatewise} v \Rightarrow \phi(u) \geq \phi(v)$$

Theorem [Cohen, Jeavons, Ž. CP'07 & TCS'08]

- $\langle \mathbf{R}_{2,1}^{\max} \rangle \subsetneq \langle \mathbf{R}_{2,2}^{\max} \rangle \subsetneq \langle \mathbf{R}_{2,3}^{\max} \rangle = \mathbf{R}_{2}^{\max}$
- $\langle \mathbf{G}_{2,1}^{\max} \rangle \subsetneq \langle \mathbf{G}_{2,2}^{\max} \rangle \subsetneq \langle \mathbf{G}_{2,3}^{\max} \rangle = \mathbf{G}_{2}^{\max}$
- $\forall d \geq 3 : \langle \mathbf{R}_{d,1}^{\max} \rangle \subsetneq \langle \mathbf{R}_{d,2}^{\max} \rangle = \mathbf{R}_{d}^{\max}$

  $\langle \mathbf{G}_{d,1}^{\max} \rangle \subsetneq \langle \mathbf{G}_{d,2}^{\max} \rangle = \mathbf{G}_{d}^{\max}$
- $\forall d \geq 2 : \langle \mathbf{F}_{f,1}^{\max} \rangle \subsetneq \langle \mathbf{F}_{f,2}^{\max} \rangle \subsetneq \langle \mathbf{F}_{f,3}^{\max} \rangle \subsetneq \langle \mathbf{F}_{f,4}^{\max} \rangle \cdots$

Multimorphism Example

- $\langle \text{Min}, \text{Max} \rangle$ is a **multimorphism** of ϕ of arity k over D iff for $\forall u, v \in D^k$, $\phi(\text{Min}(u,v)) + \phi(\text{Max}(u,v)) \ \leq \ \phi(u) + \phi(v)$
- Example: $\phi(x,y,z) = x + 2y + 3z$ over $D = \{1, \ldots, 5\}$

$$\begin{array}{lcl} u & \langle 1,2,3 \rangle & \\ v & \langle 5,1,2 \rangle & \end{array} \xrightarrow{\phi} \left.\begin{array}{r} 14 \\ 13 \end{array}\right\} \sum = 27$$

$$\ge$$

$$\begin{array}{lc} u' = \text{Min}(u,v) & \langle 1,1,2 \rangle \\ v' = \text{Max}(u,v) & \langle 5,2,3 \rangle \end{array} \xrightarrow{\phi} \left.\begin{array}{r} 9 \\ 18 \end{array}\right\} \sum = 27$$

Multimorphisms

- multimorphisms are mappings from D^k to D^k
- multimorphisms characterise several tractable valued constraint languages
- $\langle L \rangle$ is characterised by **fractional polymorphisms**, (ugly) weighted mappings from D^k to D^n
- problem: what is the simplest algebraic property which characterise the expressive power of valued constraints (and gives a Galois connection)?
  (in progress, LP & Farkas Lemma)

SFM

- minimisation of $\psi : 2^V \to \mathbb{Q}_+$ is NP-hard
- ψ submodular, $\psi(U \cup V) + \psi(U \cap V) \leq \psi(U) + \psi(V)$, $O(n^6)$
- equivalent to submodular constraints over $\{0, 1\}$
- ϕ is submodular $\Leftrightarrow$ $\langle \textsc{Min}, \textsc{Max} \rangle$ is a multimorphism of ϕ
- bounded-arity submodular constraints
- via min-cuts: binary and ternary
- problem: all bounded-arity solved by min-cuts?

Theorem [Ž., Jeavons '08, submitted]

$\forall k \geq 4$, new $L_k \subsetneq L_{sub,k}$ s.t. VCSP(L_k) solvable by min-cuts

From Constraints to Computer Vision

So far:

- VCSP framework, properties, modelling in VCSP
- language restrictions, VCSP(L)
- expressive power $\langle L \rangle$, algebraic properties

Now:

- how results on valued constraints relate to (and generalise) problems in computer vision

MRF vs. VCSP

- Boolean domain = $\{0, 1\}$ vs. binary (2-ary) constraint
- MRF (CRF) $\approx$ VCSP
- (high-order) clique of size $k \approx$ constraint of artiy k
- reducing to graph cuts, we do not express energy exactly as it is obvious (and takes extra space)
- submodularity: finite-valued & Boolean domain
  (encoding (lin/log) obvious and the rest follows)
  (more details: Schrijver, CCJK'06 (AI), JCC'98 (AI))
- "negative-positive" constraints

MRF vs. VCSP 2

- testing submodularity of arity $k \geq 4$ co-NP-C
- hence, no poly-time T s.t. f submodular iff T(f) submodular
- but we do not need that!
- given a submodular constraint of arity k,
  how to express it by binary?
- explicit bound on $\sharp$ extra variables
  (for any expressibility gadget, not only submodular)
- finding the gadget is another issue
- algebraic, non-constructive, approach

Thank you!

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